ISBN 978-93-5156-328-0
International Conference of Advance Research and Innovation (ICARI-2014)

A Study of Reliability analysis in Stochastic Dependency

Vipin Kumar [a,*], Ajeet Singh [b], Satpal Singh [a]

[a]Department of Mathematics, Teerthanker Mahaveer University, Moradabad, Uttar Pradesh, India
[b]Department of Mathematics, Kunwar Satyavira College of Engineering and Management, Bijnor, Uttar Pradesh, India

Article Info

Article history:
Received 29 December 2013
Received in revised form
10 January 2014
Accepted 20 January 2014
Available online 1 February 2014

Keywords

PQD (Positive quadrant dependent),
NQD (negative quadrant dependent),
SI (Stochastically increasing),
LTD (Left-tail decreasing),
RTI (right-tail increasing),
TP^2 (Totally positive of order 2),
PQDE (Positive quadrant dependent in expectation),
RCSI (Right corner set increasing).

Abstract

In this paper, I focus my attention on a relatively weaker notion of dependence, namely the positive quadrant dependence between two variables X and Y. I think that this easily verified form of positive dependence is more relevant in the subject area under discussion. Also, as might be expected, the notions of dependence are simpler and their relationships are more readily exposed in the bivariate case than the multivariate ones.

1. Introduction

The concept of dependence permeates the world. There are many examples of interdependence in the medicines, economic structures, and reliability engineering. Usually an example in engineering is that all output from a piece of equipment will depend on the input in a broader sense, which includes material, equipment, environment, and others. The dependence is not deterministic but of a stochastic process. In this paper, I have limited the scope of my discussion to the dependence notions that is relevant to reliability analysis. In the reliability, it is usually assumed that the component lifetimes are independent. However, components in the same system are used in the same environment or share the same load, and hence the failure of one component affects the others. I also have the type of so called common cause failure and components might fail at the same time. The dependence is usually difficult to describe, even for very similar components. From light bulbs in an overhead projector to engines in an aero plane, I have dependence, and it is essential to study the effect of dependence for better reliability design and analysis. There are many notions of bivariate and multivariate dependence. Several of these concepts were motivated from applications in reliability. As one probably would expect, they refer to the form of positive dependence between two or more variables. I shall try to explain them and their interrelationships in this paper. Positive dependence means that large values of Y tend to accompany large values of X, and vice versa for small values. Discussion of the concepts of dependence involves refining this basic idea, by means of definitions and deductions. Hutchinson and Lai [1] devoted a chapter to reviewing concepts of dependence for a bivariate distribution. Joe [2] gave a comprehensive treatment of the subject on multivariate dependence. Thus, my goal here is not to provide another review; instead, I shall focus my attention on the positive dependence, in particular the positive quadrant dependence. I confine my selves mainly to the bivariate case, although most of my discussion can be generalized to the multivariate situations. An important aspect of this paper is the availability of several examples that are employed to illustrate the concepts of positive dependence. In this paper, I am trying to develop a

Corresponding Author,
E-mail address: vipinkumar2629@gmail.com

fundamental dependence idea with the following structure:

1. positive dependence, a general concept
2. important conditions describing positive dependence
3. positive quadrant dependence—conditions and applications
4. examples of positive quadrant dependence
5. positive dependence orderings

2. Positive Dependence

Concepts of stochastic dependence for a bivariate distribution play an important part in statistics. it is often convenient to refer to the bivariate distributions that satisfy it as a family, a group. In this paper, I am mainly concerned with positive dependence. Although negative dependence concepts do exist, they are often obtained by negative analogues of positive dependence via reversing the inequity signs. Various notions of dependence are motivated from applications in statistical reliability (e.g. see Barlow and Proschan [3]). The base line or starting point, of the reliability analysis of systems is independence of the lifetimes of the components. Mostly, it is often more realistic to assume some form of positive dependence among components. In 1970, several studies discussed different notions of positive dependence between two random variables, and derived some interrelationships among them, e.g. Lehmann [4], Esary et al. [5], Esary and Proschan [6], Harris [7], and Brindley and Thompson [8], among others. Yanagimoto [9] unified some of these notions by introducing a family of concepts of positive dependence. Some further notions of positive dependence were introduced by Shaked [10–12]. For concepts of multivariate dependence, see Block and Ting [13] and a more recent text by Joe [2].

3. Basic Conditions

According to Jogdeo [14], the following basic conditions describing positive dependence; these are in increasing order of stringency (Jogdeo [15]).

1. Positive correlation, cov(X, Y) ≥ 0.
2. For every pair of increasing functions a and b, defined on the real line R, cov[a(X), b(Y)] ≥0.
 Lehmann [4] showed that this condition is equivalent to

$$\Pr(X \geq x, Y \geq y) \geq \Pr(X \geq x) \Pr(Y \geq y)$$

 Or equivalently,

$$\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x) \Pr(Y \leq y)$$

I can say that (X, Y) shows positive quadrant dependence if and only if these inequalities hold.

3. Esary et al. [5] introduced the following condition, termed "association": for every pair of functions a and b, defined on R^2, that are increasing in each of the arguments (separately) cov[a(X, Y), b(X, Y)] ≥ 0
 Here, I notice that a direct verification of this dependence concept is difficult in general.
 However, it is often easier to verify one of the alternative positive dependence notions that
 Imply association.
4. Y is right-tail increasing in X if Pr(Y > y X>x) is increasing in x for all y (written as RTI(Y | X). Similarly, Y is left-tail decreasing if Pr(Y ≤ y | X ≤ x) is decreasing in x for all y (written as LTD(Y | X)).
5. Y is said to be stochastically increasing in x for all y (written as SI(Y | X)) if, for every y, Pr(Y > y | X = x) is increasing in x. Similarly, we say that X is stochastically increasing in y for all x (written as SI(X | Y)) if for every x, Pr(X > x | Y = y) is increasing in y.
6. Let X and Y have a joint probability density function f (x, y). Then f is said to be totally positive of order two (TP^2) if for all $x_1 < x_2$, $y_1 < y_2$:

$$f(x_1, y_1) f(x_2, y_2) \geq f(x_1, y_2) f(x_2, y_1)$$

4. Positive Quadrant Dependent in Expectation

Now, I am introducing a slightly less stringent dependence notion that would include the PQD distributions. For any real number x, let Y_x be the random variable with distribution function Pr(Y ≤ y |X>x). It is easy to verify that the inequality in the conditional distribution Pr(Y ≤ y |X>x) ≤ Pr(Y ≤ y) implies an inequality in expectation $E(Y_x) \geq E(Y)$ if Y is a non-negative random variable.

I described that Y is PQDE on X if the last inequality involving expectation holds. Similarly, there is negative quadrant dependent in expectation if E(Yx) ≤ E(Y).

It is easy to show that PQD⇒PQDE by observing PQD is equivalent to Pr(Y > y |X> x) ≥ Pr(Y > y), which in turn implies $E(Y_x) \geq E(Y)$ (assuming Y ≥ 0).

$$Cov(X,Y) = \iint [\bar{F}(x,y) - \bar{F}_X(x)\bar{F}_Y(y)]dxdy$$

$$= \int \bar{F}_X(x)\{\int [\Pr(Y > y \mid X > x) - \bar{F}_Y(y)]dy\}dx$$

$$= \int \bar{F}_X(x)[E(Y_x) - E(Y)]dx$$

Which is greater than zero if X and Y are PQDE Thus, PQDE implies that cov(X, Y) ≥ 0. In other words, PQDE lies between PQD and positive correlation. There are many bivariate random variables being PQDE, because all the PQD distributions with $Y \geq 0$ are also PQDE.

5. Positive Quadrant Dependent Concept

There are many notions of bivariate dependence known in the literature. The notion of positive quadrant dependence appears to be more straightforward and easier to verify than other notions. The rest of the work mainly focuses on this dependence concept. The definition of PQD, which was first given by Lehmann [4], is now reproduced below.

Definition: Random variables X and Y are PQD if the following inequality holds:

$$\Pr(X > x, Y > y) \geq \Pr(X > x)\Pr(Y > y)$$

for all x and y. Equation 7.6 is equivalent to

$$\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x)\Pr(Y \leq y)$$

PQD is shown to be a stronger notion of dependence than the positive (Pearson) correlation but weaker than the "association", which is a key concept of positive dependence in Barlow and Proschan [3], originally introduced by Esary et al. [5].

Consider a system of two components that are arranged in series. By assuming that the two components are independent, when they are in fact PQD, I will underestimate the system reliability. For systems in parallel, on the other hand, assuming independence when components are in fact PQD, will lead to overestimation of system reliability. This is because the other component will fail earlier knowing that the first has failed. This, from a practical point of view, reduces the effectiveness of adding parallel redundancy. Thus a proper knowledge of the extent of dependence among the components in a system will enable us to obtain a more accurate estimate of the reliability characteristic in question.

6. Constructions of Positive Quadrant Dependent Bivariate Distributions

Let F(x, y) denote the distribution function of (X, Y) having continuous marginal cumulative distribution functions $F_X(x)$ and $F_Y(y)$ with marginal probability distribution functions $f_X = F_X^{'}$ *and* $f_Y = F_Y^{'}$ respectively. For a PQD bivariate distribution, the joint distribution function may be written as

$$F(x, y) = F_X(x)F_Y(y) + w(x, y)$$

satisfying the following conditions:

$$w(x, y) \geq 0$$

$$w(x,\infty)\to 0, \qquad w(\infty, y)\to 0,$$

$$w(x, -\infty) = 0, \qquad w(-\infty, y) = 0$$

$$\frac{\partial^2 w(x,y)}{\partial x \partial y} + f_X(x) f_Y(y) \geq 0$$

Note that if $X \geq 0$ and $Y \geq 0$, then the condition in above equation may be replaced by $w(x,\infty)\to 0$, $w(\infty, y)\to 0$, $w(x, 0) = 0$, $w(0, y) = 0$.

Lai and Xie [16] used these conditions to construct a family of PQD distributions with uniform marginals.

7. Negative Quadrant Dependence

Although themain theme of this chapter is on positive dependence, it is a common knowledge that negative dependence does exist in various reliability situations. The bivariate normal, FGM family, Durling–Pareto distribution, and bivariate exponential distribution of Sarmanov are NQD when the ranges of the dependence parameter are appropriately specified. The two variables of the following example can only be negatively dependent.

Example: Gumbel's bivariate exponential distribution

$$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)} \qquad 0 \leq \theta \leq 1$$

$$F(x, y) - F_X(x)F_Y(y) = e^{-(x+y+\theta xy)} - e^{-(x+y)} \leq 0 \qquad 0 \leq \theta \leq 1$$

8. Conclusion

Concepts of stochastic dependence are widely applicable in statistics. Given that some of these concepts have arisen from reliability contexts, it seems rather unfortunate that not many reliability practitioners have caught onto this important subject. This observation is transparent, since the assumption of independence is still prevailing in many reliability analyses. Among the dependence concepts, the correlation is a widely used concept in applications. Association is advocated and studied in Barlow and Proschan [3]. On the other hand, PQD is a weaker condition and also seems to be easier to verify. On reflection, this phenomenon may be due in part to the fact that many of the proposed dependence models are often not readily applicable. One would hope that, in the near future, more applied probabilists and reliability engineers would get together to forge a partnership to bridge the gap between the theory and applications of stochastic dependence.

ISBN 978-93-5156-328-0

References

[1] TP Hutchinson, CD Lai. Continuous bivariate distributions, emphasising applications. Adelaide, Australia: Rumsby Scientific Publishing; 1990.

[2] H Joe. Multivariate models and dependence concepts. London: Chapman and Hall; 1997.

[3] RE Barlow, Proschan F. Statistical theory of reliability and life testing: probability models. Silver Spring (MD): To Begin With; 1981.

[4] Lehmann EL. Some concepts of dependence. Ann Math Stat 1966; 37:1137–53.

[5] JD Esary, F Proschan, DW Walkup.Association of random variables, with applications.AnnMath Stat 1967;38:1466–74.

[6] JD Esary, F Proschan. Relationships among some bivariate dependence. Ann Math Stat 1972; 43:651–5.

[7] R Harris. A multivariate definition for increasing hazard rate distribution. Ann Math Stat 1970;41:713–7

[8] EC Brindley, WA Thompson. Dependence and aging aspects of multivariate survival. J Am Stat Assoc 1972; 67:822–30.

[9] T Yanagimoto. Families of positive random variables. Ann Inst Stat Math 1972; 559–73.

[10] M. A Shaked concept of positive dependence for exchangeable random variables. Ann Stat 1977; 5:505–15.

[11] M Shaked. Some concepts of positive dependence for bivariate interchangeable distributions. Ann Inst Stat Math 1979; 31:67–84.

[12] Shaked M. A general theory of some positive dependence notions. J Multivar Anal 1982; 12: 199–218.

[13] H W Block, M L Ting,. Some concepts of multivariate dependence. Commun Stat A: Theor Methods 1981; 10:749–62.

[14] K Jogdeo. Dependence concepts and probability inequalities. In: GP Patil, S Kotz, JK Ord, editors. A modern course on distributions in scientific work – models and structures, vol. 1. Dordrecht: Reidel; 1975. p. 271–9.

[15] K Jogdeo. Dependence concepts of. In: Encyclopedia of statistical sciences, vol. 2. New York: Wiley; 1982. p. 324–34.

[16] CD Lai, M.Xie A new family of positive dependence bivariate distributions. Stat Probab Lett 2000; 46:359–64.

[17] H. A Robbins remark on the joint distribution of cumulative sums. Ann Math Stat 1954; 25:614–6.

[18] A W. Kimball On dependent tests of significance in analysis of variance. Ann Math Stat 1951; 22:600–2.

[19] R Douglas, SE Fienberg, Lee MLT, AR Sampson, LR Whitaker. Positive dependence concepts for ordinal contingency tables. In: IMS lecture notes monograph series: topics in statistical dependence, vol. 16. Hayward (CA): Institute of Mathematical Statistics; 1990. p. 189–202.

[20] XieM, Lai, CD.On the increase of the expected lifetime by parallel redundancy. Asia–Pac J Oper Res 1996; 13:171–9.

[21] Mi J. Bolstering components for maximizing system lifetime. Nav Res Logist 1998; 45:497–509.

[22] S Kotz, CD Lai, M. Xie, The expected lifetime when adding redundancy in systems with dependent components. IIE Trans 2003; in press.

[23] Farlie DJG. The performance of some correlation coefficients for a general bivariate distribution. Biometrika 1960; 47: 307–23.

[24] N L Johnson, S. Kotz Distributions in statistics: continuous multivariate distributions. New York:Wiley; 1972.

[25] S P Mukerjee, BC Sasmal. Life distributions of coherent dependent systems. Calcutta Stat Assoc Bull 1977; 26:39–52.

[26] G S Lingappaiah. Bivariate gamma distribution as a life test model. Aplik Mat 1983; 29:182–8.

[27] M J Philips. A preventive maintenance plan for a system subject to revealed and unrevealed faults. Reliab Eng 1981; 2:221–31.

[28] S Kotz, NL Johnson. Some replacement-times distributions in two-component systems. Reliab Eng 1984; 7:151–7.

[29] KV Mardia. Families of bivariate distributions. London:Griffin; 1970.

[30] D V Lindley, N D Singpurwalla. Multivariate distributions for the life lengths of components of a system sharing a common environment. J Appl Probab.1986; 23:41831