ISBN 978-93-5156-328-0
International Conference of Advance Research and Innovation (ICARI-2014)

Examine the Parametric Linear Programming with Different Effects

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Article Info

Article history:
Received 29 December 2013
Received in revised form
10 January 2014
Accepted 20 January 2014
Available online 1 February 2014

Keywords

Linear Programming,
Parametric Linear Programming,
Simplex Method

Abstract

In this paper we will examine the effect of varying the coefficient of the objective function and then examine the effect of varying the right hand side constants.

1. Parameterizing the Objective Function

The term parametric linear programming is applied to the situation where the coefficient of the objective functions and /or the right-hand-side constants are allowed to vary with a parameter, say θ.

In this case, the objective function coefficient c_j is assumed to change simultaneously at given rates γ_j. Thus, the class of linear programs of interest is:

Minimize $(c+\theta\gamma)T_x = x(\theta)$

Subject t $Ax = b$ $A: m\times n$

$x \geq 0,$ …. (1)

Where $\gamma = (\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n)^T$ are the given fixed rates of change of the objective function coefficient $c = (c_1, c_2, c_3, ...c_n)^T$ per unit of the scalar parameter $\theta \geq 0$.We shall examine the behavior of (1) as θ varies. Without loss of generality we have assumed $\theta \geq 0$ because the case of $\theta \leq 0$ is equivalent to replacing θ by $-\bar{\theta}$.

The feasibility of problem (1) is clearly independent of the objective function; thus we shall only examine the case when the problem is feasible. Let us consider that the objective function has a finite optimum when θ=0 and the optimal basis is B. Let π be the optimal prices for a basis B when θ=0 and $\hat{\pi} = \pi + \theta\rho$ for some θ>0. Then for given value of θ, we can determine π and ρ from

$B^T\hat{\pi}(\theta) = B^T(\pi + \theta\rho) = c_B + \theta\gamma_B$ ……... (2)

i.e π and ρ are solution to

$B^T\pi = c_B$ and $B^T\rho = \gamma_B$ …….. (3)

Next we determine the reduced costs $\hat{\sigma}(\theta)$ with respect to B from

$$\hat{\sigma}(\theta) = c_N + \theta\gamma_N - N^T\hat{\pi}(\theta)$$

$$= c_N - N^T\pi + \theta(\gamma_N - N^T\rho) \quad …… (4)$$

We are interested in the range of $\theta \geq 0$ for which B is an optimal basis .in particular by the assumed optimality of B for θ=0, we have

$$\bar{c}_N = \hat{\sigma}(\theta) = c_N - N^T\pi \geq 0 \quad ...… (5)$$

The range of $\theta \geq 0$ for which B is an optimal basis is the range for which $\hat{\sigma}(\theta) \geq 0$,

From equation (4), we require

$$\hat{\sigma}(\theta) = \bar{c}_N + \theta\bar{\gamma}_N \geq 0, \quad …… (6)$$

Where $\bar{\gamma}_N = \gamma_N - N^T\rho$ ……. (7)

Then from (6), the basis B remains optimal for θ satisfies the vector relation

$-\theta\bar{\gamma}_N \leq \bar{c}_N$ Where $\bar{c}_N \geq 0$ ……… (8)

Two cases arise in determining the range of θ that maintains optimality:

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ISBN 978-93-5156-328-0
International Conference of Advance Research and Innovation (ICARI-2014)

1. F1. $\overline{\gamma}_N \geq 0$ then the basis B is optimal for all values for $\theta \geq 0$.
2. if one or more components $\overline{\gamma}_j < 0$ for $j = N$,then the basis B is optimal for all θ in the range $0 \leq \theta \leq \theta_1$ where

$$\theta_1 = \min_{\{j \in N | \overline{\gamma}_j < 0\}} \frac{\overline{c}_j}{-\overline{\gamma}_j} \qquad \text{………. (9)}$$

For $\theta = \theta_1 + \in$ where $\in > 0$ the basis B will no longer be optimal, and one or more non basic variables will become candidates for entering the basis.

For the rest of this discussion, assume that we are solving the problem in the canonical form of the Simplex Method. If $\theta > \theta_1$, then one or more non basic variables will become candidates for entering the basis B. It is clear that if any candidate variable x_s has coefficients $\overline{a}_{is} \leq 0$ for i=1, 2… m, the problem is unbounded for all $\theta > \theta$ because we have found a ray along which the objective function can be made arbitrarily small. When this happens, we terminate with the class of solutions

$x_B = b - \alpha \overline{A}_s,\ x_s = \alpha \geq 0,\ x_j = 0\ \ for\ \ j \in N\ \ and\ \ j \neq s$

where the corresponding $z \to -\infty as \alpha \to \infty$.

LEMMA 1 (Nonnegative of Relative Cost Factors)

One eligible candidate x_s to enter the basis at $\theta = \theta_1 + \in > 0$, $\in > 0$ and there are one or more coefficients $\overline{a}_{is} > 0$ then the relative cost factor with respect to the new basis at $\theta = \theta_1$ define by (8) are nonnegative and remain nonnegative for some range of $\theta > \theta_1$.

Proof. It is evident that $\overline{c}_s + \theta_1 \overline{\gamma}_s = 0$ and $\overline{c}_j + \theta_1 \overline{\gamma}_j > 0$ For $j \in N$ and $j \neq s$ because we are assuming only one candidate x_s .If we pivot on $\overline{a}_{rs}$ assuming $\theta \geq \theta_1$ .we get the new reduced costs for columns equal to zero and for the remaining non basic columns we get

$$\hat{c}_j = \overline{c}_j + \theta \overline{\gamma}_j - \frac{\overline{a}_{rj}}{\overline{a}_{rs}} (\overline{c}_s + \theta \overline{\gamma}_s),\ j \in N,\ j \neq s. \qquad \text{………………………… (10)}$$

Noting that $\overline{c}_s + \theta_1 \overline{\gamma}_s = 0$ we rewrite Equation (10) as

$$(\overline{c}_j + \theta_1 \overline{\gamma}_j) + (\theta - \theta_1)\left(\overline{\gamma}_j - \frac{\overline{a}_{rj}}{\overline{a}_{rs}} \overline{\gamma}_s\right) \qquad \text{……. (11)}$$

Because the assumption $\overline{c}_j + \theta \overline{\gamma}_j > 0$ for $j \in N,\ j \neq s$, the first term dominates the second term for some range $\theta > \theta$.

Theorem1 (When Minimizing the Optimal Value is a Continuous Piecewise Linear Concave Function)

The optimal value of the parametric objective function for the linear program (1) is a continuous piecewise linear concave function of the parameter θ .

Proof. Let $0 < \theta \leq \theta^*$ be the range of values for θ for which a finite minimum exist for the objective function. As θ increase from 0 to θ as defined by (9). The basis does not change and thus the basic feasible solution $(x_B, x_N) = 0$ does not change. Hence the objective function value changes linearly with θ in this range.

Similarly, for $\theta_1 < \theta \leq \theta_2$ there is a new basis and the objective function also change linearly with θ until the next point θ_2 where the next optimal basis change is reached. However, under non degeneracy, the slope $\gamma^T x$ with respect to θ is different beyond θ_1 because the optimal solution x^2 in the new interval is not the same as the optimal solution x^1 in the previous interval. (Under degeneracy it is possible that $x^2 = x^1$ implying the slopes are the same.) Thus, in general, the function is clearly piecewise linear and continuous.

Let θ' and θ'' be any two points in the interval $0 \leq \theta \leq \theta^*$ and let x' and x'' be the corresponding feasible optimal solutions to (1) with optimal objective function values $z(\theta')$ and $z(\theta'')$ respectively. Pick any λ in the range $0 \leq \lambda \leq 1$ and define $\theta^\lambda = \lambda\theta' + (1-\lambda)\theta''$ .Let the optimal solution at θ^λ be denoted x^λ and the optimal objective value by $z(\theta^\lambda)$ .Then

$$\begin{aligned} z(\theta^\lambda) &= (c + \theta^\lambda \gamma)^T x^\lambda \\ &= \lambda(c + \theta' \gamma)^T x^\lambda + (1-\lambda)(c + \theta'' \gamma)^T x^\lambda \\ &\geq \lambda z(\theta') + (1-\lambda) z(\theta'') \end{aligned}$$

Where the last line follows from the optimality of $z(\theta')$ and $z(\theta'')$ this proves that the function is concave and we have already shown that it is pricewise linear continuous.

Corollary 1 (when maximizing, the optimal value is a continuous piecewise linear convex function)

If the objective function of the parametric linear program defined by (1) is maximized instead of minimized, then the optimal value is a continuous piecewise linear convex function of the parameter θ .

Corollary 2 (when minimizing, the optimal value is a continuous piecewise linear convex function)

If the objective function of the parametric linear program defined by (1) is of the form $z(\theta)=(1-\theta)c^T x+\theta\gamma^T x$, the optimal value is a continuous piecewise linear convex function of the parameter θ.

2. Parameterizing the Right-Hand Side

In this case the right-hand side constant b_i is assumed to change at given rates β_j.Thus, the class of linear programs of interest is:

Minimize $c^T x = z(\phi)$

Subject to $Ax=b+\phi\beta$ $A: m\times n$

$x\geq 0$ ……… (12)

Where $\beta=(\beta_1,\beta_2,\beta_3,........\beta_m)$ are the given fixed rates of change to the right-hand side per unit of the scalar parameter ϕ.Once again, without loss of generality, we restrict $\phi\geq 0$ because looking at $\phi\leq 0$ is equivalent to replacing ϕ by $-\bar{\phi}$. It can easily be verified that if β does not lie in the range space of the coefficient matrix A, the linear program is feasible only for $\phi=0$.

EXAMPLE: 1. If $Ax=b$, $x\geq 0$ is feasible and $Ax=\beta$, $x\geq 0$ is also feasible ,show that $Ax=b+\phi\beta$ is feasible for all choices of $\phi\geq 0$ also show that if a constraint is redundant for some $\phi>0$ then it is redundant for all values of $\phi>0$.

Assume that the linear program is feasible for both b and β.Then the optimal basis B at $\phi=0$ stays feasible for the range of ϕ for some range $\phi\geq 0$, namely, ϕ Satisfying:

$$B^{-1}b+\phi B^{-1}\beta\geq 0 \quad \text{....... (13)}$$

Therefore, letting $\bar{b}=B^{-1}b$ and $\bar{\beta}=B^{-1}\beta$, the basis B remains primal feasible for all ϕ. Satisfying the vector relation:

$$-\phi\bar{B}\leq\bar{b} \quad \text{.......... (14)}$$

Example2. Show that if the optimal basic feasible is non-degenerate then B says feasible and optimal for some range [0, ϕ_1], where ϕ_1 >0.

Two cases arise in determining the range of ϕ that maintains feasibility:

1. If $\bar{\beta}\geq 0$ then the optimal basis B results in a feasible solution for all values of $\phi\geq 0$. The values of the basic variables and the objective are the only ones that change, the basic set of columns remain unchanged.
2. If, on the other hand , one or more components $\bar{B}_i<0$, then the range of ϕ that maintains feasibility is $0\leq\phi\leq\phi_1$, where

$$\phi_1=\min_{\{i|i\in B,\bar{\beta}_i<0\}}\frac{\bar{b}_i}{-\bar{\beta}_i} \quad \text{... (15)}$$

At $\phi=\phi_1+\in$ where $\in>0$, the problem is primal-infeasible but is still dual-feasible since $\bar{c}_N\geq 0$ does not depend on the right-hand side b.

3. Conclusion

With the help of these two examples we see that the effect of varying the coefficient of the objective function and the effect of varying the right hand side constants.

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