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Applications of Wavelet Packets to Wave Propagations

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Abstract
Wavelets have become a powerful tool in signal and image processing for more than two decades now. In this paper, the wave propagation equation has been represented as scaling functions discretization, wavelet discretization and wavelet packet discretization. We discretise it using periodic wavelet packets. We study the reduction of wave equation for a rod, Decoupling using Eigen values analysis and discretized these equations by using the wavelet packets. Solution of the problem in transformed wavelet domain is being closely related to the spectral analysis of wave propagation.

1. Introduction

Wavelet analysis, a new mathematical discipline has gotten attention by several branches of Engineers, Physicists and Mathematicians over past three decades. Wavelet is pioneered by Grossmann and Morlet [4], Daubechies [5] and Meyer [6]. In comparison of Fourier expansion, wavelet uses many basis functions to approximate such a function well. These properties of wavelets have led to some very successful applications within the field of signal processing. Its generalization, known as wavelet packets, is pioneered by Coifmanet.al. [7, 8, 9] and Wicker hauser [10]. Periodic wavelets are introduced by Meyer [11] and thoroughly discussed by Nielson [12]. Some important similarities between periodic wavelet packets and trigonometric system have been studied by Hess-Nielson [13]. A natural starting point of projection methods is the topic of two term connection coefficients given by Barker [14] and Lotto et al. [15]. Discretization of periodic boundary value problems (1-D Helmholtz Equation) with respect to scaling functions, wavelets and wavelet packets has been studied by Kumar et.al. [2]. Transform methods have ability to solve certain difficult ordinary and partial differential equations. Wavelet transforms are the new entrants to solve these equations and are popular with electrical and communication engineering in charactering and synthesizing the time signals.

Doyle [1] has given the wave propagation in structures and Wavelet transform is used to study vibration problem, wavelet and wavelet packet transform will be used to solve wave propagation problems. The solution of the problem is transformed wavelet domain being closely related to spectral Analysis of wave propagation.

Frequency dependent wave characteristics namely, spectrum and dispersion relations are obtained for a generalized system using Discrete Fourier Transform (DFT). These relations are the frequency variation of the wave parameters termed as wave numbers and wave speeds respectively. These parameters are essential to understand the wave mechanics in a given waveguide and are also required for Spectral Finite Element (SFE) formulation at a later stage. These parameters provide information like whether the Wave Mode is a propagating mode or a damping mode or a combination of these two. Next for a propagating mode, the nature of frequency variation of wave numbers gives information whether the mode is non-dispersive i.e. the wave retains its shape as it propagates or dispersive where the shape changes with propagation. In this section, these parameters are explained using the example of a generalized one-dimensional second and fourth order system.

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1.1 Definition: A Multi-Resolution Analysis (MRA) consists of a nested sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ satisfying the following properties

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(iii) For some $f \in L^2(\mathbb{R})$, $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$.

(iv) For some $\phi \in V_0$, $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

(v) There exists an isomorphism $I : V_0 \to \ell^2(\mathbb{Z})$.

The function ϕ whose existence is implied by condition (iv) is called scaling function of the given MRA. Since $V_0 \subset V_1$ any function in V_0 can be expanded in terms of basis of functions of V_1. In particular $\phi(x) = \phi_{0,0}(x) \in V_0$, so

$$\phi(x) = \sum_{k=-\infty}^{\infty} a_k\, \phi_{1,k}(x) = \sqrt{2}\sum_{k=-\infty}^{\infty} a_k\, \phi(2x-k) \qquad (1)$$

where

$$a_k = \int_{-\infty}^{\infty} \phi(x)\, \phi_{1,k}(x)\, dx \qquad (2)$$

For completely compactly supported scaling functions only finitely many a_k will be non–zeroand we have

$$\phi(x) = \sqrt{2}\sum_{k=0}^{D-1} a_k\, \phi(2x-k) \qquad (3)$$

This equation is known as dilation Equation. D is an even positive integer called the wavelet genus and the numbers $a_0, a_1, a_2 ... a_{D-1}$ are called the filter coefficients.

In analogy to equation (1.2) we can write a relation for the basis wavelet ψ . Since $\psi \in W_0$ and $W_0 \subset V_1$ can be expand as

$$\psi(x) = \sqrt{2}\sum_{k=0}^{D-1} b_k \, \phi(2x-k) \quad (4)$$

where the filter coefficients are

$$b_k = \int_{-\infty}^{\infty} \psi(x)\, \phi_{1,k}(x)\, dx \quad (5)$$

This equation is known as wavelet equation. Since the set $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0 by axioms of MRA. It follows by repeated application of axioms of MRA that $\{\phi(2^j x - k)\}_{k\in Z}$ is an orthogonal basis for V_j and $\{2^{j/2}\phi(2^j x-k)\}_{k\in Z}$ is an orthonormal basis for V_j then there exists a function $\psi_{jk}(x)$ i.e. $\{2^{j/2}\psi(2^j x-k)\}_{k\in Z}$ is an orthonormal basis for W_j .

The further decomposition of W_j space into two orthogonal spaces whose direct sum is W_j. The functions of these two spaces are being of better frequency localization then the original ones. If $U_j^0 = V_j$ and $U_j^1 = W_j,\ j \in \mathbb{Z}$, then

$$U_{j+1}^n = U_j^{2n} \oplus U_j^{2n+1},\ j \in \mathbb{Z} \quad (6)$$

In most practical applications such as image processing, data fitting or problems involving differential equations, the space domain is a finite interval. Many of these cases can be dealt with by introducing periodized scaling functions, wavelets and wavelet packets.

1.2 Definition: Let $\phi \in L^2(\mathbb{R})$ be the scaling function, then 1-periodic scaling function for $j, l \in Z$ is defined as

$$\tilde{\phi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n) = 2^{j/2}\sum_{n=-\infty}^{\infty} \phi(2^j(x+n)-l), \quad x \in \mathbb{R} \quad (7)$$

Let $\psi \in L^2(\mathbb{R})$ be the basic wavelet, then the 1-periodic wavelet is defined as

$$\tilde{\psi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \psi_{j,l}(x+n) = 2^{j/2}\sum_{n=-\infty}^{\infty} \psi(2^j(x+n)-l), \quad x \in \mathbb{R} \quad (8)$$

If $\omega_n \in L^2(\mathbb{R})$ be the basic wavelet packet then the 1-periodic wavelet packet is defined as

$$\tilde{\omega}_{n,j,l}(x) = \sum_{p=-\infty}^{\infty} \omega_{n,j,l}(x+p) = 2^{j/2}\sum_{p=-\infty}^{\infty} \omega_n(2^j(x+p)-l), \quad x \in \mathbb{R} \quad (9)$$

1.1 Connection Coefficients: The abstract connection coefficients of the orthonormal bases are given by Barker [14], Lattoet. al.[15], Perrier and Wickerhauser [17] and Kunth [18]. We define the connection coefficients as

$$\Gamma_{j,l,m}^{d_1,d_2} = \int_{-\infty}^{\infty} \phi_{j,l}^{(d_1)}(x)\ \phi_{j,l}^{(d_2)}(x)\, dx,\ j,\ l,\ m \in \mathbb{Z}$$

where d_1 and d_2 are orders of differentiations. Let us consider that these derivatives are well-defined. Using the change of variable $x \leftarrow (2^j x - l)$, we obtain

$$\Gamma_{j,l,m}^{d_1,d_2} = 2^{jd}\int_{-\infty}^{\infty} \phi^{(d_1)}(x)\ \phi^{(d_2)}(x-m+l)\, dx = 2^{jd}\ \Gamma_{0,0,m-l}^{d_1,d_2},$$ where $d_1 + d_2 = d$.

Repeated application of integration by parts yields the identity $\Gamma_{0,0,n}^{d_1,d_2} = (-1)^{d_1}\Gamma_{0,0,n}^{0,d}$ because the scaling functions have compact support. Hence

$$\Gamma_{j,l,m}^{d_1,d_2} = (-1)^{d_1}\ 2^{jd}\ \Gamma_{0,0,m-l}^{0,d}$$

1.2 Wavelet Packet Expansion of a function $f \in L^2(\mathbb{R})$: If a function $f \in L^2(\mathbb{R})$ then

$$f(x) = \sum_{k=0}^{2^{J_0}-1} c_{J_0,k}\ \tilde{\phi}_{J_0,k}(x) + \sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^j-1} d_{n,l,k}\ \tilde{\omega}_{n,l,k}(x) \quad (10)$$

where $l = j - P$, $P = J_0, J_{0+1}, J_{0+2} ..., J$

and $d_{n,l,k}$ the wavelet packet coefficients is defined as

$$d_{n,l,k} = \langle f,\ \tilde{\omega}_{n,l,k}\rangle = \int_{-\infty}^{\infty} f(x)\,\tilde{\omega}_{n,l,k}(x)\ dx \quad (11)$$

Now from equation (10), we have

$$f(x) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\ \tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\ \tilde{\omega}_{n,l,k}(x) \quad (12)$$

where

$$c_{J_0,l} = \int_{-\infty}^{\infty} f(x)\,\tilde{\phi}_{J_0,l}(x)\, dx \quad (13)$$

and

$$d_{n,l,k} = \int_{-\infty}^{\infty} f(x)\,\tilde{\omega}_{n,l,k}(x)\ dx \quad (14)$$

1.3 Expansion of Periodic Functions: Let $f \in \tilde{V}_J$ and J_0 satisfy $0 \le J_0 \le J$. The decomposition $\tilde{V}_J = \tilde{V}_{J_0} \oplus \left(\bigoplus_{j=J_0}^{J-1} \tilde{W}_j\right)$, which is obtain from $\tilde{V}_J \oplus \tilde{W}_J = \tilde{V}_{J+1}$ leads to two expansions of f namely the pure periodic scaling function expansion

$$f(x) = \sum_{l=0}^{2^J-1} c_{J,l}\ \tilde{\phi}_{J,l}(x),\quad x \in [0,\ 1] \quad (15)$$

and the periodic wavelet expansion

$$f(x) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\tilde{\psi}_{j,l}(x),\quad x \in [0,1] \quad (16)$$

If $J_0 = 0$, than the equation (1.16) becomes

$$f(x) = c_{0,0} + \sum_{j=J_0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\ \tilde{\psi}_{j,l}(x) \quad (17)$$

Now periodic wavelet packets expansion be

$$f(x) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\ \tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\ \tilde{\omega}_{n,l,k}(x),\ x \in [0,1] \quad (18)$$

Now we define the periodic extension $\tilde{f}$ of f as

$$\tilde{f}(x) = f(x - \lfloor x \rfloor),\ x \in \mathbb{R} \quad (19)$$

Then 1-periodicity of $\tilde{f}$ can be verified as

$$\tilde{f}(x+1) = f(x+1-(\lfloor x+1 \rfloor)) = f(x - \lfloor x \rfloor) = \tilde{f}(x),\ x \in \mathbb{R} \quad (20)$$

As $\lfloor x \rfloor$ is an integer, we have $\tilde{\phi}(x - \lfloor x \rfloor) = \tilde{\phi}(x),\ \tilde{\psi}(x - \lfloor x \rfloor) = \tilde{\psi}(x)$ and $\tilde{\omega}_n(x - \lfloor x \rfloor) = \tilde{\omega}_n(x),$ for $x \in \mathbb{R}$. Equation (20) applying (18) gives

$$\tilde{f}(x) = f(x - \lfloor x \rfloor) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\ \tilde{\phi}_{J_0,l}(x - \lfloor x \rfloor) + \sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\ \tilde{\omega}_{n,l,k}(x - \lfloor x \rfloor)$$

$$= \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\ \tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\ \tilde{\omega}_{n,l,k}(x),\ x \in \mathbb{R} \quad (21)$$

The coefficients in (15), (16) and (18) are respectively given by

$$c_{j,l} = \int_{-\infty}^{\infty} f(x)\tilde{\phi}_{j,l}(x)\, dx;\ d_{j,l} = \int_{-\infty}^{\infty} f(x)\tilde{\psi}_{j,l}(x)\, dx;$$

$$d_{n,l,k} = \int_{-\infty}^{\infty} f(x)\,\tilde{\omega}_{n,l,k}(x)\ dx$$

but in fact these coefficients are same as those of the non-periodic expansions. To prove this we use the fact that $\tilde{f}(x) = f(x),\ x \in [0,\ 1]$ and we have

$$d_{n,l,k} = \int_0^1 \tilde{f}(x)\tilde{\omega}_{n,l,k}(x)\,dx = \sum_{n=-\infty}^{\infty}\int_0^1 \tilde{f}(x)\omega_{n,l,k}(x+n)\,dx$$
$$= \int_{-\infty}^{\infty} \tilde{f}(y)\omega_{n,l,k}(y)dy \qquad (22)$$

1.4 Differentiating Matrix with Respect to Scaling Function: Let $f$be a function in $V_J \cap C^d(\mathbb{R})$ and $J \in N_0$ differentiating both sides of equation $f(x) = \sum_{l=-\infty}^{\infty} c_{J,l}\,\phi_{J,l}(x)$; $x \in \mathbb{R}$, d–times, we obtain

$$f^{(d)}(x) = \sum_{l=-\infty}^{\infty} c_{J,l}\,\phi_{J,l}^{(d)}(x)\ ; \qquad x \in \mathbb{R} \qquad (23)$$

It should be noted that $f^{(d)}(x)$ in general not belong to V_j; so projection of $f^{(d)}$ back onto V_j be

$$\left(P_{V_j} f^{(d)}\right)(x) = \sum_{k=-\infty}^{\infty} c_{J,k}^{(d)}\phi_{J,k}(x), \qquad x \in \mathbb{R} \qquad (24)$$

where accordingly, we have

$$c_{J,k}^{(d)} = \int_{-\infty}^{\infty} f^{(d)}(x)\phi_{J,k}(x)dx \qquad (25)$$

Substituting equation (1.23) in equation (1.25), we get

$$c_{J,k}^{(d)} = \sum_{l=-\infty}^{\infty} c_{J,l}\int_{-\infty}^{\infty}\phi_{J,k}(x)\phi_{J,l}^{(d)}(x)dx \Rightarrow c_{J,k}^{(d)} = \sum_{l=-\infty}^{\infty} c_{J,l}\,\Gamma_{J,k,l}^{0,d}$$
$$\Rightarrow c_{J,k}^{(d)} = \sum_{l=-\infty}^{\infty} c_{J,l}\,2^{Jd}\Gamma_{l-k}^{d} \Rightarrow c_{J,k}^{(d)} = \sum_{n=-\infty}^{\infty} c_{J,n+k}\,2^{Jd}\Gamma_n^d, \quad -\infty < k < \infty$$

We used $\Gamma_{J,l,m}^{d_1 d_2} = (-1)^d\,2^{Jd}\,\Gamma_{m-l}^d$ for the Last equality. Since Γ_n^d is only non-zerofor $n \in [2-D, D-2]$, so we define

$$c_{J,k}^{(d)} = \sum_{n=2-D}^{D-2} c_{J,n+k}\,2^{Jd}\,\Gamma_n^d\ ;\ J,k \in \mathbb{Z} \qquad (26)$$

Recall that if f is 1–periodic then

$$c_{J,l} = c_{J,l+P2^J}, \qquad l,P \in \mathbb{Z} \text{ and } c_{J,k}^{(d)} = c_{J,k+P2^J}^{(d)} \qquad k,P \in \mathbb{Z}$$

Hence it is sufficient to consider 2^J coefficient of either type or equation (1.26) becomes

$$c_{J,k}^{(d)} = \sum_{n=2-D}^{D-2} c_{J,\langle n+k\rangle_{2^J}}\ 2^{Jd}\,\Gamma_n^d \ ;\ k = 0,1,\ldots,2^J-1 \qquad (27)$$

This system of equations can be represented in matrix form (vector form)

$$c^{(d)} = D^{(d)}c \qquad (28)$$

where $\left|D^{(d)}\right|_{k\langle n+k\rangle_{2^J}} = 2^{Jd}\,\Gamma_n^d$,

$k = 0,1,\ldots,2^J-1;\ n = 2-D\ldots\ldots, D-2$,

and $c^{(d)} = \left[c_{J,0}^{(d)}, c_{J,1}^{(d)}, \cdots, c_{J,2^J-1}^{(d)}\right]$.

We will refer the matrix $D^{(d)}$ as the differentiation matrix of order d. It can be seen that $D^{(d)}$is symmetric for d is even and skew symmetric for d is odd.

2. Wave Propagation Equations

The generalized second-order partial differential Equation givenby $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} = c\frac{\partial^2 u}{\partial t^2}$ (29)

wherea, b, c are unknown constants depending on the material properties and geometry of the waveguide,$u(x,t)$ is the field variable to be solved for with x being the spatial dimension and t the temporal dimension. First $u(x,t)$ is transformed to frequency domain using DFT as

$$u(x,t) = \sum_{n=0}^{N-1} \hat{u}_n(x,\omega_n)e^{i\omega_n t} \qquad (30)$$

whereω_n is the discrete circular frequency in rad/sec and N is the total number of frequency points used in the transformation. Here ω_n is related to the time window by

$$\omega_n = n\Delta\omega = \frac{n\omega_f}{N} = \frac{n}{N\Delta t} = \frac{n}{T}$$

whereΔt is the time sampling rate and ω_f is the highest frequency captured by Δt. The frequency content of the load decides N and consideration of the *warp around* and *aliasing problem* decides $\Delta\omega$. Here $\hat{u}_n$ is the n^{th} DFT coefficient and can also be referred to as the coefficient at frequency ω_n. $\hat{u}_n$ varies only with$x$substituting equation (30) into equation (29) we get $a\frac{d^2\hat{u}_n}{dx^2} + b\frac{d\hat{u}_n}{dx} + c\omega_n^2\hat{u}_n = 0, n = 0,1,2,3,\ldots,N-1$ (31)

or $a\ \hat{u}_n''(x) + b\ \hat{u}_n'(x) + c\omega_n^2\hat{u}_n(x) = 0$ (32)

and$\hat{u}_n(x) = \hat{u}_n(x+1)$ (33)

wherea, b, c and $\omega_n^2 \epsilon\ \mathbb{R}$. Thus, through DFT the governing PDE given by equation (29 is reduced to the N ODE varying only in x.

2.1.Representation with respect to scaling functions: The discretization of equation (32) by $\hat{u}_n(x)$ be

$$\hat{u}_{nJ}(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\hat{u}_n}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} \quad J\epsilon\ N_o \ (34)$$

Following the approach that leads the equation (24), we find

$$\hat{u}_{nJ}''(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\hat{u}_n}^{(2)}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} (35)$$

where $\left(C_{\hat{u}_n}^{(2)}\right)_{Jk}$is given by

$$\left(C_{\hat{u}_n}^{(2)}\right)_{Jk} = \left[D^{(2)}C_{\hat{u}_n}\ \right]_{Jk} = \sum_{n=2-D}^{D-2}\left[C_{\hat{u}_n}\ \right]_{J(n+k)2^J} 2^{Jd}\,\Gamma_n^d, k$$
$$= 0,1,2,\ldots,2^J-1$$

We can use the Galerkin Method to determine the coefficients $(C_{\hat{u}_n})_{Jk}$. Multiplying equation (32) by $\tilde{\varphi}_{Jk}(x)$ and integrating over the limit [0,1], we get

$$a\int_0^1 \hat{u}_{nJ}''(x)\,\tilde{\varphi}_{Jk}(x)dx + b\int_0^1 \hat{u}_{nJ}'(x)\,\tilde{\varphi}_{Jk}(x)dx + c\omega_n^2\int_0^1 \hat{u}_{nJ}(x)\,\tilde{\varphi}_{Jk}(x)dx = 0$$

Using equation (34), (35) and orthogonality of the periodized scaling functions we get $a\left[C_{\hat{u}_n}^{(2)}\ \right]_{Jk} + b\left[C_{\hat{u}_n}^{(1)}\ \right]_{Jk} + c\omega_n^2\left[C_{\hat{u}_n}\right]_{Jk} = 0,\ \ k = 0,1,2,3,\ldots,2^J-1$ (36)

In vector notation this becomes

$$aC_{\hat{u}_n}^{(2)} + bC_{\hat{u}_n}^{(1)} + c\omega_n^2 C_{\hat{u}_n} = 0 \qquad (37)$$

andusing the approach of the equation (1.28), we can get a linear system of equations

$$MC_{\hat{u}_n} = 0 \qquad (38)$$

where

$$M = aD^{(2)} + bD^{(1)} + c\omega_n^2 I \quad (39)$$

Here D is given by the equation (2.11). For simplicity we can replace $D^{(2)}$ by $D^2$and $D^{(1)}$ by D and obtain

$$MC_{\hat{u}_n} = 0 \qquad (40)$$

where

$$M = aD^2 + bD + c\omega_n^2 I \qquad (41)$$

Equations (2.10) and (2.12) represent the scaling function discretization of equation (2.4).

2.2. Representation with respect to Wavelets: Taking equation (2.10) as a point of departure and using the relation$d_{\hat{u}_n} = WC_{\hat{u}_n}$, we get

$MW^T d_{\hat{u}_n} = 0$ (42)

Let $\breve{M} = WMW^T$ then

$\breve{M} = W(aD^{(2)} + bD^{(1)} + c\omega_n^2 I)W^T$ or $\breve{M} = a\breve{D}^{(2)} + b\breve{D}^{(1)} + c\omega_n^2 I$ (43)

where $\breve{D}^{(2)}$ and $\breve{D}^{(1)}$ are defined as in following equation

$\breve{D}^{(1)} = WD^{(1)}W^T$ and $\breve{D}^{(2)} = WD^{(2)}W^T$

From equation (42), we get

$WMW^T d_{\hat{u}_n} = 0 \Rightarrow \breve{M} d_{\hat{u}_n} = 0$ (44)

This is the wavelet discretization of equation (32).

2.3.Representation with respect to Wavelet packets: Taking equation (38) as a point of departure and using the relation $d_{\hat{u}_n} = \omega_n C_{\hat{u}_n}$ we get

$M\omega_n^T d_{\hat{u}_n} = 0$ (45)

Let us define $M^* = \omega_n M \omega_n^T$, then

$M^* = \omega_n\left(aD^{(2)} + bD^{(1)} + c\omega_n^2 I\right)\omega_n^T$ or $M^* = aD^{*(2)} + bD^{*(1)} + C\omega_n^2 I$ (46)

where $D^{*(2)}$ and $D^{*(1)}$ are defined as in following equation

$D^{*(2)} = \omega_n D^{(2)}\omega_n^T$ and $D^{*(1)} = \omega_n D^{(1)}\omega_n^T$

From equation (2.17), we get

$\omega_n M\omega_n^T d_{\hat{u}_n} = 0 \Rightarrow M^* d_{\hat{u}_n} = 0$ (47)

This is wavelet packet discretization of equation(32).

3. Reduction of Wave Equation (For A Rod)

The governing differential Wave Equation of an isotopic rod is given as

$$EA\frac{\partial^2 u}{\partial x^2} - \eta A\frac{\partial u}{\partial t} = \rho A\frac{\partial^2 u}{\partial t^2} \quad (48)$$

where E, A, η and ρ are the young's modulus, Cross-sectional area, damping ratio and density respectively. $u(x,t)$ is the axial deformation. Let $u(x,t)$ be discretized at n points in the time window $[0, t_f]$. Let $\tau = 0,1,2,\ldots,n-1$ be the sampling points then

$$t = \Delta t\, \tau \quad (49)$$

Where, Δt is the time interval between two sampling points. The function $u(x,t)$ can be approximated by scaling function $\varphi(\tau)$ at an arbitrary scale as

$$u(x,t) = u(x,\tau) = \sum_k u_k(x)\varphi(\tau-k);\quad k\epsilon Z \quad (50)$$

where $u_k(x)$ are the approximation coefficients at a certain spatial location x. Substituting Equation (49) and (50) in equation (3.1), we get

$$EA\sum_k \frac{d^2u_k}{dx^2}\varphi(\tau-k) - \frac{\eta A}{\Delta t}\sum_k u_k\varphi'(\tau-k) = \frac{\rho A}{\Delta t^2}\sum_k u_k\varphi''(\tau-k) \quad (51)$$

Taking inner product both sides of (51) with $\varphi(\tau-j)$ where $j = 0,1,2,\ldots,n-1$, we get

$$EA\sum_k \frac{d^2u_k}{dx^2}\int \varphi(\tau-k)\varphi(\tau-j)d\tau - \frac{\eta A}{\Delta t}\sum_k u_k\int \varphi'(\tau-k)\varphi(\tau-j)d\tau = \frac{\rho A}{\Delta t^2}\sum_k u_k\int \varphi''(\tau-k)\,\varphi(\tau-j)d\tau \quad (52)$$

These translates of scaling functions are orthogonal i.e.

$$\int \varphi(\tau-k)\varphi(\tau-j)d\tau = 0 \quad \text{for } j \neq k \quad (53)$$

Using equation (53), equation (52) can be written as n simultaneous ODES

$$EA\frac{d^2u_j}{dx^2} - \frac{\eta A}{\Delta t}\sum_{k=j-N+2}^{j+N-2}\Omega_{j-k}^1 u_k = \frac{\rho A}{\Delta t^2}\sum_{k=j-N+2}^{j+N-2}\Omega_{j-k}^2 u_k, j = 0,1,2,\ldots,n-1 \quad (54)$$

$$\Rightarrow EA\frac{d^2u_j}{dx^2} = \sum_{k=j-N+2}^{j+N-2}\left(\frac{\eta A}{\Delta t}\Omega_{j-k}^1 + \frac{\rho A}{\Delta t^2}\Omega_{j-k}^2\right)u_k, j = 0,1,2,\ldots,n-1 \quad (55)$$

where N is the order of the Daubechies wavelet and Ω_{j-k}^1 and Ω_{j-k}^2 are the connection coefficientsgiven by

$\Omega_{j-k}^1 = \int \varphi'(\tau-k)\varphi(\tau-j)d\tau$ and $\Omega_{j-k}^2 = \int \varphi''(\tau-k)\varphi(\tau-j)d\tau$

For the Daubechiescompactlysupported wavelets Ω_{j-k}^1 and Ω_{j-k}^2 are non-zero only in the interval $k = j-N+2$ to $k = j+N-2$. The details of evaluation of connection coefficients for different order are given by Beylkin[3].

The forced boundary condition associated with governing differential equation given by equation (49) is

$$EA\frac{\partial u}{\partial x} = F \quad (56)$$

where $F(x,t)$ is the axial forced applied and can be approximated similar as $u(x,t)$ in equation (3.3) as

$$F(x,t) = F(x,\tau) = \sum_k F_K(x)\varphi(\tau-k), k\epsilon Z \quad (57)$$

Substituting equation (3.3) and (3.10) in equation (3.9) and taking the inner product with $\varphi(\tau-j)$ we get

$$EA\frac{d\,u_j}{dx} = F_j \quad, j = 0,1,2,\ldots,n-1 \quad (58)$$

$$\text{or}\quad \left.\begin{aligned} EAu_j'(x) &= F_j(x)\\ u_j(x) &= u_j(x+1)\end{aligned}\right\}, x\,\epsilon\,\mathbb{R}$$

where $EA\,\epsilon\,\mathbb{R}$ and $F_j(x) = F_j(x+1)$.

3.1.Representation with respect to scaling functions: The discretization of equation (60) by $u_j(x)$ replaced by the approximation

$$u_{jJ}(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{u_j}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\}\quad J\epsilon N_o \quad (59)$$

Following the approach that leads the equation (24), we find

$$u_{jJ}'(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{u_j}^{(1)}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} \quad (60)$$

where $\left(C_{u_j}^{(1)}\right)_{Jk}$ are defined by using the approach as in equation (27)and given by

$$\left(C_{u_j}^{(1)}\right)_{Jk} = \left[D^{(1)}C_{u_j}\right]_{Jk} = \sum_{n=2-D}^{D-2}[C_{u_j}]_{J(n+k)2^J}2^{Jd}\Gamma_n^d,\ k = 0,1,2,\ldots,2^J-1$$

By using Galerkin Method one can determine the coefficients $\left[C_{u_j}\right]_{Jk}$. Multiplying equation (3.12) by $\tilde{\varphi}_{Jk}(x)$ and integrating over the limit [0,1], we get

$$EA\int_0^1 u_{jJ}'(x)\,\tilde{\varphi}_{Jk}(x)dx = \int_0^1 F_j(x)\,\tilde{\varphi}_{Jk}(x)dx$$

Using equations (60), (61) and orthogonality of periodized scaling functions, we get

$$EA\left(C_{u_j}^{(1)}\right)_{Jk} = \left(C_{F_j}\right)_{Jk},\quad k = 0,1,2,3,\ldots,2^J-1 \quad (61)$$

where

$$[C_{F_j}]_{Jk} = \int_0^1 F_j(x)\,\tilde{\varphi}_{Jk}(x)dx \quad (62)$$

In vector notation this becomes

$EAC_{u_j}^{(1)} = C_{F_j}$ (63)

andusing the approach of the equation (28), we can arrive a linear system of equations

$BC_{u_j} = C_{F_j}$ (64)

where

$B = EAD^{(1)}$ (65)

Alternatively we can replace $D^{(1)}$ by D, where $D^{(1)}$ is given by the equation$D^{(1)} = D$ and obtain

$BC_{u_j} = C_{F_j}$ (66)

where

$B = EAD$ (67)

Equations (65), (67) represent the scaling function discretization of equation (59).

3.2. Representation with respect to Wavelets: Taking equation (65) as a point of departure and using the relations$d_{u_j} = W \ C_{u_j}$and$d_{F_j} = W \ C_{F_j}$, we get

$BW^T d_{u_j} = W^T d_{F_j}$ (68)

Let $\breve{B} = WBW^T$ then $\breve{B} = W(EAD^{(1)})W^T or$ $\breve{B} = EA\breve{D}^{(1)}$ (69)

where$\breve{D}^{(1)}$ is defined as in following equation $\breve{D}^{(1)} = WD^{(1)}W^T$.

From equation (68), we get

$WBW^T d_{u_j} = WW^T d_{F_j} \Rightarrow \breve{B} d_{u_j} = d_{F_j}$ (70)

This is the wavelet discretization of equation (59).

3.3. Representation with respect to Wavelet packets: Taking equation (65) as a point of departure and using the relations$d_{u_j} = \omega_n C_{u_j}$and $d_{F_j} = \omega_n C_{F_j}$, we get

$B\omega_n^T d_{u_j} = \omega_n^T d_{F_j}$ (71)

Let $B^* = \omega_n B \omega_n^T$, then

$B^* = \omega_n (EAD^{(1)}) \omega_n^T$ or$B^* = EAD^{*(1)}$ (72)

where $D^{*(1)}$ is defined as in following equation $D^{*(1)} = \omega_n D^{(1)} \omega_n^T$.

Then from equation (71), we get

$\omega_n B \omega_n^T d_{u_j} = \omega_n \omega_n^T d_{F_j} \Rightarrow B^* d_{u_j} = d_{F_j}$ (73)

This is the wavelet packet discretization of equation ().

For handling finite length data sequences the coefficient u_j near the vicinity of the boundaries ($j = 0$ and $j = n-1$) should be treated appropriately. Similar to the solution of ordinary differential equations for structural dynamics problem the wavelet extrapolation technique is used here to treat the boundary coefficients. After treating the boundaries for the finite system, equation (55) can be written in matrix form

$$\left\{\frac{d^2 u_j}{dx^2}\right\} = \left(\frac{\eta A}{EA}\Gamma^1 + \frac{\rho A}{EA}\Gamma^2\right)\{u_j\} \quad (74)$$

It should be noted that thought all formulations are done with reference to the governing differential equation for a rod the connection coefficient matrices Γ^1 and$\Gamma^2$are independent of the problem and depend only on the order of wavelet or wavelet packet *i.e.*, N.

4. Decoupling using Eigen Value Analysis

It can be seen from the above derivations that the wavelet coefficients of first and second derivatives can be obtained as

$\{\dot{u}_j\} = \Gamma^1\{u_j\}$ (75)

$\{\ddot{u}_j\} = \Gamma^2\{u_j\}$ (76)

The second derivative can also be written as

$\{\ddot{u}_j\} = \Gamma^1\{\dot{u}_j\}$ (77)

On substituting equation (4.1) in equation (4.3), we get

$\{\ddot{u}_j\} = [\Gamma^1]^2\{u_j\}$ (78)

Thus through the second order connection coefficient matrix Γ^2can be evaluated independently as

$\Gamma^2 = [\Gamma^1]^2$ (79)

The above modification is done as this form helps in imposing the initial conditions for non-periodic solution. Thus the Equation (74) can be written as

$$\left\{\frac{d^2 u_j}{dx^2}\right\} = \left(\frac{\eta A}{EA}\ [\Gamma^1] + \frac{\rho A}{EA}[\Gamma^1]^2\right)\{u_j\} \quad (80)$$

In wavelet Spectral Finite Element (WSFE) the reduced ODEs are coupled however the system of equation can be decoupled by diagonalizing the connection coefficient matrixΓ^1. This can be done by Eigenvalue analysis of the matrix as

$\Gamma^1 = \varphi\pi\varphi^{-1}$ (81)

whereφ is the Eigen matrix of$\Gamma^1$and π is the diagonal matrix containing corresponding eigenvalues $-i\gamma_j$. From Equation (79),Γ^2can be written as

$\Gamma^2 = \varphi\pi^2\varphi^{-1}$ (82)

where π^2 is the diagonal matrix with diagonal term $-\gamma_j^2$. This Eigenvalue analysis is costly but can be done once and stored as it completely independent of the problem. This makes the computational time comparable to Fourier transform based methods.

ODEs obtained by decoupling the Equation (4.6) can be written as

$$\frac{d^2 \hat{u}_j}{dx^2} = -\left(\frac{\eta A}{EA}\, i\gamma_j + \frac{\rho A}{EA}\gamma_j^2\right)\hat{u}_j,\ \ j = 0,1,2,\dots,n-1 \quad (83)$$

where

$\hat{u}_j = \varphi^{-1} u_j$ (84)

Similarly the force boundary condition given by equation (3.11) can be written as

$$\frac{d\hat{u}_j}{dx} = \hat{F}_j\ ,\ \ j = 0,1,2,\dots,n-1 \quad (85)$$

where

$EA\hat{F}_j = \varphi^{-1}\hat{F}_j$ (86)

Or now the equation (4.11) can be written as

$$\left.\begin{array}{l}\hat{u}_j'(x) = \hat{F}_j\ \ (x)\\ \hat{u}_j(x) = \hat{u}_j(x+1)\end{array}\right\};\quad x \in \mathbb{R} \quad (87)$$

where $\hat{F}_j(x) = \hat{F}_j(x+1)$.

4.1.Representation with respect to scaling functions: The discretization of equation (87) by $\hat{u}_j\ (x)$ replaced by the approximation

$$\hat{u}_{jJ}(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\hat{u}_j}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} \quad J \epsilon\, N_o \ (88)$$

Following the approach that leads the equation (24), we find

$$\hat{u}_{jJ}'(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\hat{u}_j}^{(1)}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} (89)$$

where$\left(C_{\hat{u}_j}^{(1)}\right)_{Jk}$ are defined by using the approach as in equation (27) and given by

$$\left(C_{\hat{u}_j}^{(1)}\right)_{Jk} = \left[D^{(1)}C_{\hat{u}_j}\right]_{Jk} = \sum_{n=2-D}^{D-2}[C_{\hat{u}_j}]_{J(n+k)2^J}2^{Jd}\,\Gamma_n^d, k = 0,1,2,\dots,2^J-1 \ (90)$$

We can use the Galerkin Method to determine the coefficients $[C_{\hat{u}j}]_{Jk}$. Multiplying equation (87) by $\tilde{\varphi}_{Jk}(x)$and integrating over the limit [0,1], we get

$$\int_0^1 \hat{u}_{jJ}'(x)\,\tilde{\varphi}_{Jk}(x)dx = \int_0^1 \hat{F}_j\ \ (x)\,\tilde{\varphi}_{Jk}(x)dx$$

Using equation (88), (89) and orthogonality of periodized scaling functions we get

$\left(C^{(1)}_{\hat{u}_j}\right)_{Jk} = \left(C_{\hat{F}_j}\right)_{Jk}, \quad k = 0,1,2,3\ldots,2^J - 1$ (91)

where

$$[C_{\hat{F}_j}]_{Jk} = \int_0^1 \hat{F}_j(x)\tilde{\varphi}_{Jk}(x)dx \qquad (92)$$

In vector notation we have

$C^{(1)}_{\hat{u}_j} = C_{\hat{F}_j}$ (93)

andusing equation (28), we can get a linear system of equations

$$BC_{\hat{u}_j} = C_{\hat{F}_j} \qquad (94)$$

where

$$B = D^{(1)} \qquad (95)$$

On replacing $D^{(1)}$ by D where D is given by the equation $D^{(1)} = D$, we obtain

$$BC_{\hat{u}_j} = C_{\hat{F}_j} \qquad (96)$$

where

$$B = D \qquad (97)$$

Equations (4.20), (4.22) represent the scaling function discretization of equation (4.13).

4.2. Representation with respect to Wavelets: Taking equation (4.20) as a point of departure and using the relations$d_{\hat{u}_j} = W \; C_{\hat{u}_j}$and $d_{\hat{F}_j} = W \; C_{\hat{F}_j}$, we get

$$BW^T d_{\hat{u}_j} = W^T d_{\hat{F}_j} \qquad (98)$$

Let us define $\breve{B} = WBW^T$ then

$\breve{B} = W(D^{(1)})W^T \Rightarrow \breve{B} = \breve{D}^{(1)}$ (99)

where$\breve{D}^{(1)}$ is defined as in following equation $D^{(1)} = WD^{(1)}W^T$.

Then from equation (98), we get

$WBW^T d_{\hat{u}_j} = WW^T d_{\hat{F}_j} \Rightarrow \breve{B} d_{\hat{u}_j} = d_{\hat{F}_j}$ (100)

This is the wavelet discretization of equation (87).

4.3. Representation with respect to Wavelet packets: Taking equation (94) as a point of departure and using the relations$d_{\hat{u}_j} = \omega_n C_{\hat{u}_j}$and $d_{\hat{F}_j} = \omega_n C_{\hat{F}_j}$, we get

$B\omega_n^T d_{\hat{u}_j} = \omega_n^T d_{\hat{F}_j}$ (101)

Let us define$B^* = \omega_n B \omega_n^T$ then

$$B^* = \omega_n D^{(1)} \omega_n^T \text{or} B^* = D^{*(1)} \qquad (102)$$

where$D^{*(1)}$ is defined as in following equation $D^{*(1)} = \omega_n D^{(1)} \omega_n^T$.

Then from equation (101), we get

$\omega_n B \omega_n^T d_{\hat{u}_j} = \omega_n \omega_n^T d_{\hat{F}_j} \Rightarrow B^* d_{\hat{u}_j} = d_{\hat{F}_j}$ (103)

This is wavelet packet discretization of equation (87).

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