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Tensor Analysis and its Applications in Physics

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Abstract
Tensor Analysis has their many applications in most all the disciplines of science. This paper, gives the basic history of tensor calculus and in sequence and study about the applications of tensor calculus in theory of relativity, elasticity and physics.

1. Introduction

The notion of tensors is as the extension of vectors. The scalar and vector quantities do not cover completely many physical and geometrical quantities. For instance, stress in an elastic body and curl of a vector supposedly misunderstood as vectors are much more than vectors. Indeed, they are tensors.

The great Mathematician *Gregorio Ricci* was first introduced and studied about the tensors, so G. Ricci is known as a founder of tensor calculus. The emergence of tensor calculus, otherwise known as the absolute differential calculus, as a systematic branch of mathematics is due to *Ricci* and his pupil *Levi - Civita*. In collaboration they published the first memoir on this subject :

The tensor formulation was originated by *G. Ricci* and it become rather popular when *Albert Einstein* used it as a natural tool for the description of his general theory of relativity. The concept of a tensor has its origin in the developments of differential geometry by Gauss, Riemann and Christoffel.

A tensor is a system of quantities or functions whose components obey a certain law of transformation of coordinates from one system to another. Generally, tensors are three types; one is contravariant tensor, second is covariant tensor, and third is mixed tensor. The rank or order of a tensor is defined as the total number of indices of the components of a tensor. A tensor of rank zero is a scalar. The examples of scalars are mass, time, distance, speed, temperature, energy etc., which are invariant under changes of the coordinate system. A tensor of rank one is a vector. The examples of vectors are displacement, velocity, acceleration, force, electric field etc. The velocity of a fluid at any point is a contra variant tensor of rank one. Also, the tangent vector to a curve is a contra variant tensor of rank one.

The normal to a surface
$f(x\, i) = c$ (Constant) is a covariant tensor of rank one.

A tensor of the second rank is the next in order of complexity after scalars and vectors. By a second rank tensor is meant a quantity uniquely specified by nine real numbers in 3 – dimensional space. The stress tensor, the moment of inertia tensor, the deformation tensor etc. are the examples of the second rank tensor. In n - dimensional space, it is specified by n^2 functions. The Kronecker delta is a mixed tensor of rank two.

2. Application of Tensor Calculus

In the course of study, we find that the nature of a tensor quantity does not change by the change of coordinate system. Due to this property, it is very useful for describing the physical laws mathematically and so it is of great use in general Relativity theory, Differential geometry, Riemannian geometry, Mechanics, Elasticity, hydrodynamics, electromagnetic theory and many other disciplines of Science and engineering.

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The laws of physics can not depend on the frame of reference which the physicist chooses for the purpose of description. Accordingly it is aesthetically desirable and often convenient to utilise the tensor calculus as the mathematical background in which such laws can be formula-ted. In particular, Einstein found it an excellent tool for the presentation of general relativity theory. By use of tensor calculus we can observe that the space is flat or not. For example; if the Riemannian curvature R^h_{ijk} or $R_{hijk} = g_{ha} R^a_{ijk}, (g_{ha} \neq 0)$ is zero, then the space is flat.

The application of tensor calculus are discussed here by taking one by one the following topics of application areas:

3. Tensor Calculus and Theory of Relativity

The general theory of Relativity which was developed by Einstein in order to disuss gravitation. He postulated the principle of covariance, which asserts that the laws of physics must be independent of the space - time coordinates. This swept away the privileged role of the Lorentz transformation. As a result of Minkowski space was replaced by a 4 - dimensional Riemannian space (V_4) with the general metric

$$ds^2 = g_{ij}\, dx^i dx^j. \tag{1}$$

Einstein also introduced the principle of equivalence, which in essence states that the fundamental tensor g_{ij} can be chosen to account for the presence of a gravitational field. That is, g_{ij} depends on the distribution of matter and energy in physical space. Matter and energy can be specified by the energy - momentum tensor T^{ij} which in the special theory satisfies the equation $T^{ij}_{,i} = F^j$. The only forces, namely those due to gravitation, are however already taken into account by the choice of the fundamental tensor g_{ij}. We therefore ignore F^j and, in accordance with the principle of covariance, the energy - momentum tensor must now satisfy the equation $T^{ij}_{,i} = 0$. We shall write this equation in the equivalent form $T^{ij}_{,i} = 0$ where $T^i_{,j} = g_{jk} T^{ik}$ is the mixed energy - momentum tensor. The problem now is to determine $T^i_{,j}$ as a function of the fundamental tensor g_{ij} and their derivatives up to the second order, bearing in mind that $T^i_{,j,i} = 0$.

The Einstein tensor $G^i_{,j}$ plays a very fundamental role in general theory of relativity. The Einstein tensor is defined as

$$G^i_{,j} = g^{ik} R_{jk} - \frac{1}{2} R\, \delta^i_j \tag{2}$$

satisfies the equation $G^{i}_{,j,i} = 0.$ The equations of motion require $T^{i}_{,j,i} = 0,$ but very remarkably $G^{i}_{,j,i} = 0$ is an identity in *Riemannian geometry*. This led Einstein to propose the relation

$$\chi T^{i}_{,j} + G^{i}_{,j} = 0. \tag{3}$$

In effect these equations form the link between the physical energy - momentum tensor $T^{i}_{,j}$ and the geometrical tensor $G^{i}_{,j}$ of the M_4 of general relativity. In order that Newton's theory of gravitation can be deduced as a first approximation from Einstein theory, it was found necessary to choose $\chi = 8\pi k/c^4$ where *k* is the usual gravitational constant $6.664 \times 10^{-8} cm.^3 gm.^{-1} sec.^{-2}$. The value of *c* is $2.99796 \times 10^{10} cm.sec.^{-1}$, and χ is $2.073 \times 10^{-48} cm.^{-1} gm.^{-1} sec.^2$ in *c. g. s.* units.

The motion of a particle which moves under the action of some force system can be represented in Minkowski space of curve, called the world- line of the particle. In special theory, the world - lines of free particles and of light rays are respectively the geodesics and the null - geodesics of Minkowski space. The principle of equivalence demands that all particles be regarded as free particles when gravitation is the only force under consideration. Then it follows from the principle of covariance that the world - line of a particle under the action of gravita-tional forces is a geodesic of the V_4 with the metric (1). If no forces act on the particle, then the world - line of a free particle is a geodesic of the Minkowski space. Similarly, the world - line of a light ray is a null - geodesic.

The velocity of a light ray is the constant c, and so we see from the relation $d\sigma^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2(dx^4)^2$ that for such a ray $d\sigma = 0.$ accordingly, the world - line of a light ray is a null geodesic of the Minkowski space.

Also, the study of planetary motion, Einstein's field equations, poisson's equations as an approximation of Einstein's field equations, Einstein's universe and De Sitter's universe etc. become so simple and interesting by use of tensor calculus. Thus, tensor calculus provide the tool for description of the general theory of *Relativity*.

4. Tensor Calculus and Elasticity

The forces acting on a body are either external or internal. The external forces may consist either of body forces such as gravity which act on every particle of it, or of surface forces which act on the external surface of the body, for example the pressure between two bodies in contact. When an elastic body is subjected to an external force or stress, it becomes deformed or strained. In the formulation of the equations governing the equilibrium of an elastic solid the *Cartesian tensor* of second rank are used. A *Cartesian tensor* is defined as-

A *Cartesian tensor* of the n^{th} order in a 3 - dimensional Euclidean space is defined as a set of 3^n quantities which transform according to equation (2.8) when the coordinates undergo a positive orthogonal transformation. This is a less straingent condition than that imposed on a tensor. So we see that all tensors are *Cartesian tensor* but a *Cartesian tensor* is not necessarily a tensor in the usual sence. A tensor $A_{k_1 k_2 \dots k_n}$ is a *Cartesian tensor* of the n^{th} order if the transformed com-ponents satisfy

$$\bar{A}_{i_1 i_2 \dots i_n} = A_{k_1 k_2 \dots k_n} a_{i_1 k_1} a_{i_2 k_2} \dots a_{i_n k_n}. \tag{4}$$

on change of the coordinates by the positive orthogonal transformation $\bar{y}_i = a_{ij} y_j$. We see from $\bar{y}_i = a_{ij} y_j$ that both y_j and their differentials dy_i are *Cartesian vectors*. Also, the *Koneker delta* is a *Cartesian tensor* of the second order because

$$\bar{\delta}_{ij} = a_{ir} a_{js} \delta_{rs} = a_{ir} a_{jr} = \delta_{ij}$$

in virtue of $a_{ji} a_{ki} = \delta_{jk}$. Similarly, a permutation tensor e_{ijk} is a *Cartesian tensor* of the third order.

The fundamental tensor of the Euclidean space is the *Koneker delta* δ_{ij}. Hence all the Christoffel symbols are zero and so the comma notation for covariant derivatives now denotes the familiar partial derivatives which are *Cartesian tensors*.

The study of elasticity in terms of tensors falls into three parts: a description of the strain or deformation of the elastic substance, a description of the force or stress which produces the deformation, and a generalization of Hook's law in terms of tensor form. Deformation of a body may be described by giving the change in the relative position of the parts of the body when the body is subjected to some external force. As the result of this force the configuration of the body changes and we say that the body is in a strained state.

Let us consider a point P of the body, at the position $\boldsymbol{r} = (x_1, y_1, z_1)$ relative to some fixed origin O. Let there is another point Q, at the position $\boldsymbol{r} + \delta\boldsymbol{r} = (x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1)$ or , $(x_i + \delta x_i)$. In the unstrained state the coordinates of Q relative to P are δx_i. Let each point of the body is subjected to an external force which causes displacement $\boldsymbol{u}(\boldsymbol{r})$ varying from point to point. Thus the point P is shifted to P_1 with the displacement $\boldsymbol{u}(\boldsymbol{r})$ and point Q is shifted to Q_1 with the displacement $\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{r} + \delta\boldsymbol{r})$. The coordinates of Q_1 relative to P_1 are $\delta y_i = \delta x_i + \delta u_i$ (see in figure 1). Using Taylor's expansion-

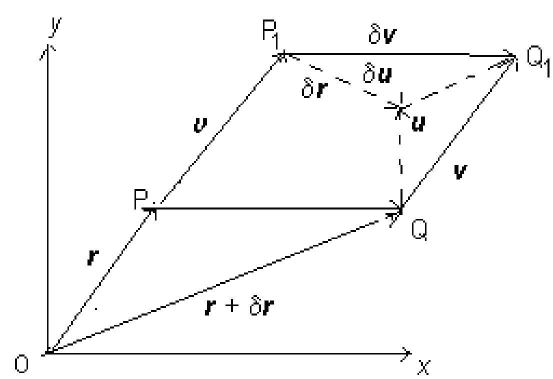

Fig.1: The point P and Q and its shifting point P₁ and Q₁ Related each other

$$\boldsymbol{u}(\boldsymbol{r} + \delta\boldsymbol{r}) = \boldsymbol{u}_i\left(\boldsymbol{x}_i + \delta\boldsymbol{x}_i\right) = \boldsymbol{u}_i\left(\boldsymbol{x}_i\right) + \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_i}\delta\boldsymbol{x}_j + \dots \tag{4.1}$$

Neglecting the second and higher order differentials, the above equation reduces to

$$\boldsymbol{u}(\boldsymbol{r} + \delta\boldsymbol{r}) = \boldsymbol{u}_i\left(\boldsymbol{x}_i\right) + \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_i}\delta\boldsymbol{x}_j = \boldsymbol{u}(\boldsymbol{r}) + grad\ \boldsymbol{u}.\delta\boldsymbol{r} \tag{4.2}$$

which shows that the displacement of the point Q consists of two parts, one of which, that is, $\boldsymbol{u}(\boldsymbol{r})$ is same for all the points of the

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body and therefore, corresponds to a translation of the body as a whole. Relative displacement Q_1 and P_1 (that is, deformation) is

$$\left.\begin{aligned} &\delta \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{r}) = (\delta\boldsymbol{r}'\, grad)\,\boldsymbol{u}. \\ \text{or,}\quad &\delta \boldsymbol{u}_i = \boldsymbol{u}_i(\boldsymbol{x}_i+\delta\boldsymbol{x}_i) - \boldsymbol{u}_i(\boldsymbol{x}_i) = \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\,\delta\boldsymbol{x}_j\,. \end{aligned}\right\} \quad (5)$$

The quantity $\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}$, being the gradient of a vector is a dyadic, that is, a tensor of second rank. Now,

$$\delta \boldsymbol{u}_i = \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\delta\boldsymbol{x}_j = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}\right)\delta\boldsymbol{x}_j + \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j} - \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}\right)\delta\boldsymbol{x}_j$$

$$= \eta_{ij}\delta\boldsymbol{x}_j + \xi_{ij}\delta\boldsymbol{x}_j, \quad (6)$$

where $\eta_{ij} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}\right)$ and $\xi_{ij} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j} - \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}\right)$ (7)

are symmetric and ant - symmetric respectively.

The term $\xi_{ij}\delta\boldsymbol{x}_j$ of the equation (6) is due to the rigid body rotation of the element. Therefore the quantity ξ_{ij} is known as *rotation tensor*. The remaining symmetric part η_{ij} is taken as a pure strain known as *strain tensor*.

The *stress* is defined as the internal force per unit area acting on a deformed body. The force in the x_i − direction acting on the face dA whose normal in the x_j − direction is $P_{ij}\,dA$, where $P_{ij}{}'s$ are actually pressures in the sence of force / area. Whenever the term force is used, it is understood that the P_{ij} are to be multiplied by the appropriate area. There are the forces acting on the small parallelopiped see in figure 2. The *stress* may be directed normally or tangentially to the surfaces on which they act. In case deforming forces act normally to a given area of an elastic medium, they produce pure alongation and the stresses are called the *normal stresses*. If the deforming forces act tangentially to the surface they produce shearing stresses.

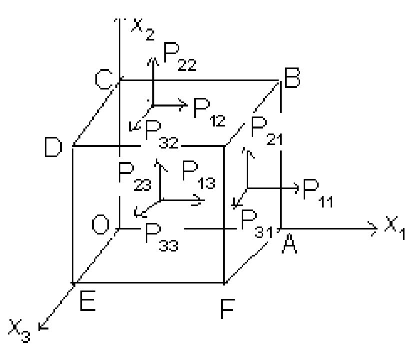

Fig.2 Forces acting on parallelepiped

Then obviously $P_{11} = \frac{\partial F_1}{\partial A_1}$, $P_{22} = \frac{\partial F_2}{\partial A_2}$, and $P_{33} = \frac{\partial F_3}{\partial A_3}$ are the normal stresses for $x_2 - x_3, x_1 - x_3$ and $x_1 - x_2$ faces respectively, where the component F_1 of the total force F act normal to the face area $A_1 = dx_2\,dx_3$ and so on. The tangential stresses $P_{12}, P_{21}, P_{13}, P_{31}, P_{23}, P_{32}$ are the shearing stresses. Thus the total stress may be completely specified by the array matrix

$$\left[P_{ij}\right] = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}, \quad (8)$$

where the diagonal elements are normal stresses and the non-diagonal elements are shearing stresses.

Let us assume that the stresses are homogeneous. Then the forces on the opposite faces will be reversed in sign as shown in figure 3.

$P_{12} = P_{21}, P_{13} = P_{31}$ and $P_{32} = P_{23}$ or, in general $P_{ij} = P_{ji}$. (9)

and hence the array of stresses, given equation (9) is symmetric. The total force along x_i' is

$$\sum_m \sum_k a_{im}\, a_{jk} P_{mk}\, da = P'_{ij}\, da \quad (10)$$

for static equilibrium. Thus in summation convention, we have

$$P'_{ij} = a_{im}\, a_{jk} P_{mk} \quad (11)$$

which is the law of transformation of a tensor of second rank as defined by equation $T'_{ik} = \sum_{j,\ell=1}^{3} a_{ij}\, a_{k\ell} T_{i\ell}$. Therefore, the array (9), with the condition (10) represents a symmetric tensor of second rank. This tensor is defined as *stress tensor*.

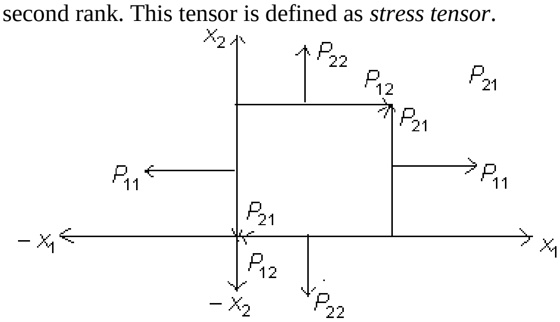

Fig.3 Forces on the opposite faces will be reversed in sign

In the elementary theory of elasticity, Hooke's law states that the tension of an elastic string is proportional to the extension. In other words, stress is proportional to strain. The corresponding assumption in the general theory of elasticity is that the stress tensor is a linear homogeneous function of the strain tensor. That is

$$P_{ij} = c_{ijk\ell}\, e_{k\ell}\,.$$

It follows from the quotient that $c_{ijk\ell}$ is a *Cartesian tensor* of the fourth order, and it is called the *elasticity tensor*. Further from the symmetry of P_{ij} and $e_{k\ell}$, we find that $c_{ijk\ell}$ is symmetric not only with respect to the indices *i* and *j* but also with respect to k and ℓ.

A body is said to be homogeneous if the elastic properties of the body are independent of the point under consideration. This means that the components of the *elasticity tensor* are all constants for a homogeneous body. We call a body isotropic if the elastic properties at a point are the same in all directions at that point. This means that the *elasticity tensor* $c_{ijk\ell}$ transforms to $c_{ijk\ell}$ itself under any rotation of axes. A *Cartesian tensor* which transforms into itself under a rotation of axes is called an *isotropic tensor*.

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5. Tensor Calculus and Physics

Tensor analyses have developed as the natural generalization of vector analysis, and therefore, it may be expected that tensor analysis has relatively vider field of applications. As pointed out earlier, the general theory of relativity was developed by using metric tensor, Christoffel symbols, curvature tensor, Ricci tensor, and Einstein tensor. By use of electromagnetic field tensor, inertia tensor, stress tensor, fundamental tensors etc., the readers feel interest in the study of physics and its relative field like dynamics and electrodynamics. The classical theory of electrodynamics, according to Lorentz, is specified by the electric potential φ which is a scalar and the magnetic potential A_i which is a vector. The electric field strength vector E_i and the magnetic field strength vector H_i are derived from these potentials by the equations

$$E_i = -\operatorname{grad}\Phi - \frac{1}{c}\frac{\partial A_i}{\partial t} \text{ and } H_i = \operatorname{curl} A_i .$$

Using electrostatic units, Maxwell's equations are

$$\left.\begin{aligned} &div\, E_i = 4\pi\rho, \\ &div\, H_i = 0, \\ &\operatorname{curl} E_i + \frac{1}{c}\frac{\partial H_i}{\partial t} = 0, \\ &\operatorname{curl} H_i - \frac{1}{c}\frac{\partial E_i}{\partial t} = \frac{4\pi}{c} j_i, \end{aligned}\right\} \quad (12)$$

where j_i is the current density vector and ρ is the charge density.

Skew - symmetrical tensor $\eta_{\alpha\beta}$ defined by

$$\eta_{\alpha\beta} = \Phi_{\alpha,\,\beta} - \Phi_{\beta,\,\alpha} = \frac{\partial \Phi_\alpha}{\partial x^\beta} - \frac{\partial \Phi_\beta}{\partial x^\alpha}$$

and we immediately calculate that its non - vanishing components in the given coordinate system are

$$\eta_{23} = -\eta_{32} = -H_1;\ \eta_{31} = -\eta_{13} = -H_2;\ \eta_{12} = -\eta_{21} = -H_3;$$

$$\eta_{14} = -\eta_{41} = -c\,E_1;\ \eta_{24} = -\eta_{42} = -c\,E_2;\ \eta_{34} = -\eta_{43} = -c\,E_3 .$$

The non - vanishing contra variant components $\eta^{\alpha\beta}$ may now be obtained and which are

$$\eta^{23} = -\eta^{32} = -H_1;\ \eta^{31} = -\eta^{13} = -H_2;\ \eta^{12} = -\eta^{21} = -H_3;$$

$$\eta^{14} = -\eta^{41} = -\frac{1}{c}E_1;\ \eta^{24} = -\eta^{42} = -\frac{1}{c}E_2;\ \eta^{34} = -\eta^{43} = -\frac{1}{c}E_3 .$$

We now write Maxwell's equations (12) in terms of η and J and the results are readily verified to be respectively

$$\left.\begin{aligned} &\frac{\partial\eta^{14}}{\partial x^1} + \frac{\partial\eta^{24}}{\partial x^2} + \frac{\partial\eta^{34}}{\partial x^3} = \frac{4\pi}{c}J^4, \\ &\frac{\partial\eta_{23}}{\partial x^1} + \frac{\partial\eta_{31}}{\partial x^2} + \frac{\partial\eta_{12}}{\partial x^3} = 0, \\ &\frac{\partial\eta_{ij}}{\partial x^4} + \frac{\partial\eta_{j4}}{\partial x^i} + \frac{\partial\eta_{4i}}{\partial x^j} = 0, \\ &\frac{\partial\eta^{1i}}{\partial x^1} + \frac{\partial\eta^{2i}}{\partial x^2} + \frac{\partial\eta^{3i}}{\partial x^3} + \frac{\partial\eta^{4i}}{\partial x^4} = \frac{4\pi}{c}J^i. \end{aligned}\right\} \quad (13)$$

The first and last equations of the relation (13) combine together into the form

$$\eta^{\alpha\beta}_{\ ,\beta} = \frac{4\pi}{c}J^\alpha, \quad (14)$$

whilst the remaining two of the relation (13) are accounted for by the equations of the set

$$\eta_{\alpha\beta,\gamma} + \eta_{\beta\gamma,\alpha} + \eta_{\gamma\alpha,\beta} = 0 \quad (15)$$

which do not vanish identically.

Maxwell's equations in tensor form in Minkowski space. Thus they are invariant under the Lorentz group of transformations.

Position of a moving particle P be determine by a vector $\boldsymbol{r}$. If the curvilinear coordi-nates of the terminal point of $\boldsymbol{r}$ are denoted by $x^i(t)$, then the equations of the path *C* of the particle can be written in the form

$$C : x^i = x^i(t), \quad (16)$$

and we call the curve *C*, the trajectory of the particle.

Velocity of a particle P is a vector $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$, whose components are

$$v^i = \frac{dx^i}{dt} \quad (17)$$

as the velocity of the particle.

In transformed coordinates the components of the velocity are

$$v'^i = \frac{dx'^i}{dt} = \frac{dx'^i}{dx^j}\frac{dx^j}{dt} = v^j\frac{dx'^i}{dx^j}, \text{ using (17)}$$

which follows the tensor law of transformation and hence the velocity of a particle at any point is a contravariant tensor (or, a contravariant vector) of rank one.

The acceleration of a particle P is a vector $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}$, and in the general case, velocity is the function of of time *t*. Let us write its intrinsic derivative with respect to *t* according to the equation

$$\frac{\delta A^\lambda}{\delta t} = \frac{dA^\lambda}{dt} + \begin{Bmatrix}\lambda \\ \alpha\beta\end{Bmatrix} A^\alpha \frac{dx^\beta}{dt}. \text{ Then}$$

$$a^i = \frac{\delta v^i}{\delta t} = \frac{dv^i}{dt} + \begin{Bmatrix} i \\ jk\end{Bmatrix} v^j \frac{dx^k}{dt} = \frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk\end{Bmatrix}\frac{dx^j}{dt}\frac{dx^k}{dt}, \quad (18)$$

Where, $\frac{\delta v^i}{\delta t}$ is the intrinsic derivative and $\begin{Bmatrix} i \\ jk\end{Bmatrix}$ are the Christoffel symbols calculated from metric tensor g_{ij}. In the cartesian rectangular coordinates the equation (17) reduces to $a^i = \frac{d^2x^i}{dt^2}$, which is obviously the acceleration of the particle. Therefore, the quantity a^i given by the equation (18) is called acceleration vector in curvilinear coordinates system. If *m* is the mass of the particle P, then the force vector (contravariant) in curvilinear coordinate is defined as

$$F^i = ma^i = m\frac{d^2x^i}{dt^2} + m\begin{Bmatrix} i \\ jk\end{Bmatrix}\frac{dx^j}{dt}\frac{dx^k}{dt}. \quad (19)$$

The covariant vector associated with F^λ may be readily be written from equation (2.24), that is,

$$F_i = g_{i\lambda} F^\lambda \quad (20)$$

and

$$F^i = g^{i\lambda} F_\lambda . \quad (21)$$

The quantity $\delta W = F_i \delta x_i$, is scalar accordingly to quotient rule. This quantity is defined as work done when the point of application of covariant force is moved through a small displacement δx_i. In cartesian coordinate system there is no distinction between covariant and contravariant components and hence $\delta W = F_i \delta x_i = F^i \delta x^i$.

From the study it clear that the tensor calculus are used so much in physics. Thus the relations in between tensor calculus and physics are much wide and by use tensor calculus, the physics becomes so simple and interesting for readers.

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