Mode to mode energy transfer in magneto hydrodynamic turbulence

Madan Lal, Swati Agarwal
Department of Applied Mathematics, Mahatma Jyotiba Phule Rohilkhand University, Bareilly (UP)

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Abstract

In the present paper, the various types of energy transfer among Fourier modes of velocity to magnetic field (**v** to **b**), velocity to velocity (**v** to **v**), magnetic field to velocity (**b** to **v**) and magnetic field to magnetic field (**b** to **b**) are considered. Using this formalism of mode to mode energy transfer, energy cascade rates (also called energy fluxes) and shell to shell energy transfer in magneto hydrodynamic turbulence has obtained. The mode to mode energy transfer in magneto hydrodynamic turbulence is computed analytically and the results obtained from the studies of flux and shell to shell energy transfers are consistent with each other.

1. Introduction

There are various phenomenological, numerical and theoretical results on magneto hydrodynamic (MHD) turbulence. Kraichnan [5] and Iroshnikov [3] proposed a first phenomenology for homogeneous and isotropic MHD which connects energy spectrum and cascade rates. Frick and Sokoloff [2] studied the spectra and cascade rates in a shell model of MHD turbulence. In MHD the invariant quantities are the total energy, the magnetic helicity and the cross helicity, whereas in fluids only energy is conserved. The properties of energy fluxes in fluid turbulence have been studied in great details starting from Kolmogrov [4] and Lesieur[6], this subject has not received much attention in MHD turbulence. The energy transfer among the velocity modes has been studied by many authors [1] and [7]. They have computed energy fluxes in two-dimensional MHD turbulence using numerical simulations. Frick and Sokoloff [2] evaluated these fluxes using shell model. Shell to shell energy transfer have been completed in fluid turbulence by Zhou [10].

In this paper, the mode to mode energy transfer in MHD turbulence have been computed analytically. The mode to mode energy transfer have been analyzed in details for turbulent velocity and magnetic field by solving the incompressible MHD equations. In incompressible MHD turbulence four energy exchanges can be identified between two wave number shells, these energy transfers are **v** to **v**, **v** to **b**, **b** to **v**, and **b** to **b**.

2. Mode to mode energy transfer in MHD turbulence

The incompressible MHD equations are

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}.\nabla)\mathbf{v} = -\frac{\nabla \mathrm{P}}{\rho}+(\mathbf{b}.\nabla)\mathbf{b}+\nu\nabla^2\mathbf{v} \qquad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t}+(\mathbf{v}.\nabla)\boldsymbol{b} = (\mathbf{b}.\nabla)\mathbf{v}+\eta\nabla^2\mathbf{v} \qquad (2)$$

$$\nabla.\mathbf{v}=0, \qquad (3)$$

$$\nabla.\mathbf{b}=0. \qquad (4)$$

Where, **v** ,**b**,P represent the velocity, magnetic and total pressure (thermal + magnetic) fields. ν and η represent kinematic viscosity and magnetic diffusivity. The equations (3) and (4) represent incompressibility condition.

In Fourier space, the kinetic energy and magnetic energy evolution equations are

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E^{v}(\alpha)=\sum_{\alpha+\beta+\gamma=0}\left\{-i(\alpha.\mathrm{v}(\gamma))(\mathrm{v}(\alpha).\mathrm{v}(\beta))+i(\alpha.\mathrm{b}(\gamma))(\mathrm{v}(\alpha).\mathrm{b}(\beta))\right\} \qquad (5)$$

and

$$\left(\frac{\partial}{\partial t}+2\eta k^2\right)E^{b}(\alpha)=\sum_{\alpha+\beta+\gamma=0}\left\{-i(\alpha.\mathrm{v}(\gamma))(\mathrm{b}(\alpha).\mathrm{b}(\beta))+i(\alpha.\mathrm{b}(\gamma))(\mathrm{b}(\alpha).\mathrm{b}(\beta))\right\} \qquad (6)$$

Equations (5) and (6) can also be written as

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E^{v}(\alpha)=\sum_{\alpha+\beta+\gamma=0}S^{vv}(\alpha|\beta|\gamma)+\sum_{\alpha+\beta+\gamma=0}S^{vb}(\alpha|\beta|\gamma) \qquad (7)$$

$$\left(\frac{\partial}{\partial t}+2\eta k^2\right)E^{b}(\boldsymbol{\alpha})=\sum_{\alpha+\beta+\gamma=0}S^{bb}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})+\sum_{\alpha+\beta+\gamma=0}S^{bv}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma}) \qquad (8)$$

If we consider an ideal case where kinematic viscosity and magnetic diffusivity are zero, i.e., (ν=0) and (η = 0).

$$\left(\frac{\partial}{\partial t}\right)E^{v}(\boldsymbol{\alpha})=\sum_{\alpha+\beta+\gamma=0}S^{vv}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})+\sum_{\alpha+\beta+\gamma=0}S^{vb}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma}) \qquad (9)$$

And

$$\left(\frac{\partial}{\partial t}\right)E^{b}(\boldsymbol{\alpha})=\sum_{\alpha+\beta+\gamma=0}S^{bb}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})+\sum_{\alpha+\beta+\gamma=0}S^{bv}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma}) \qquad (10)$$

$$S^{vv}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma}) = -i(\boldsymbol{\alpha}.\mathrm{v}(\boldsymbol{\gamma}))(\mathbf{v}(\boldsymbol{\alpha}).\mathbf{v}(\boldsymbol{\beta})) \qquad (11)$$

$$S^{vb}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma}) = i(\boldsymbol{\alpha}.\mathrm{b}(\boldsymbol{\gamma}))(\mathbf{v}(\boldsymbol{\alpha}).\mathbf{b}(\boldsymbol{\beta})) \qquad (12)$$

$$S^{bb}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma}) = -i(\boldsymbol{\alpha}.\mathrm{v}(\boldsymbol{\gamma}))(\mathbf{b}(\boldsymbol{\alpha}).\mathbf{b}(\boldsymbol{\beta})) \qquad (13)$$

$$S^{bv}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma}) = i(\boldsymbol{\alpha}.\mathrm{b}(\boldsymbol{\gamma}))(\mathbf{b}(\boldsymbol{\alpha}).\mathbf{v}(\boldsymbol{\beta})) \qquad (14)$$

Where each $SXY(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})$ term represents the mode to mode energy transfer rate from the mode $\boldsymbol{\beta}$ of the field Y into the mode $\boldsymbol{\alpha}$ of the field X, with the mode $\boldsymbol{\gamma}$ acting as a mediator. Where X and Y can be either **v** or **b**. The wavenumber triad $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ should satisfy the condition $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}=0$. Each $SXY(\boldsymbol{\alpha}|\boldsymbol{\beta},\boldsymbol{\gamma})$ term represents the energy transfer rate from the modes $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ to the mode $\boldsymbol{\alpha}$ as given below

$$S^{vv}(\boldsymbol{\alpha}|\boldsymbol{\beta},\boldsymbol{\gamma}) = S^{vv}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})+S^{vv}(\boldsymbol{\alpha}|\boldsymbol{\gamma}|\boldsymbol{\beta}) \qquad (15)$$

$$S^{vb}(\boldsymbol{\alpha}|\boldsymbol{\beta},\boldsymbol{\gamma}) = S^{vb}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})+S^{vb}(\boldsymbol{\alpha}|\boldsymbol{\gamma}|\boldsymbol{\beta}) \qquad (16)$$

$$S^{bb}(\boldsymbol{\alpha}|\boldsymbol{\beta},\boldsymbol{\gamma}) = S^{bb}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})+S^{bb}(\boldsymbol{\alpha}|\boldsymbol{\gamma}|\boldsymbol{\beta}) \qquad (17)$$

$$S^{bv}(\boldsymbol{\alpha}|\boldsymbol{\beta},\boldsymbol{\gamma}) = S^{bv}(\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\gamma})+S^{bv}(\boldsymbol{\alpha}|\boldsymbol{\gamma}|\boldsymbol{\beta}) \qquad (18)$$

Where $E\mathbf{v}(\boldsymbol{\alpha})=(\mathbf{v}(\boldsymbol{\alpha}).\mathbf{v}*(\boldsymbol{\alpha}))/2 = |\mathbf{v}(\boldsymbol{\alpha})|2/2$ is the kinetic energy; $\mathbf{v}*(\boldsymbol{\alpha})=\mathbf{v}(-\boldsymbol{\alpha})$ and $E\boldsymbol{b}(\boldsymbol{\alpha})=(\mathbf{b}(\boldsymbol{\alpha}).\mathbf{b}*(\boldsymbol{\alpha}))/2 = |\mathbf{b}(\boldsymbol{\alpha})|2/2$ is the magnetic energy ; $\mathbf{b}*(\boldsymbol{\alpha})=\mathbf{b}(-\boldsymbol{\alpha})$.

***Corresponding Author,**
E-mail address: madan.mjpru@gmail.com
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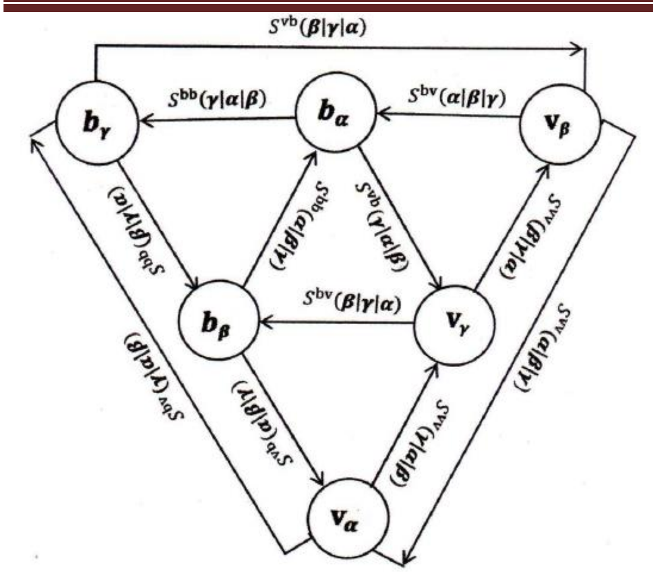


Fig.1: The mode to mode energy transfer in Magneto hydrodynamic turbulence.

The equations (11) to (14) satisfy the following properties (1) Energy transfer rate from Y to X is equal and opposite to that from X to Y, i.e.,

$$S^{XY}(\alpha|\beta|\gamma) = - S^{YX}(\beta|\alpha|\gamma) \quad (19)$$

(2) The sum of all energy transfer rates along v-v and b-b are zero, i.e.

$$S^{XX}(\alpha|\beta|\gamma) + S^{XX}(\alpha|\gamma|\beta) + S^{XX}(\beta|\alpha|\gamma) + S^{XX}(\beta|\gamma|\alpha) + S^{XX}(\gamma|\alpha|\beta) + S^{XX}(\gamma|\beta|\alpha) = 0. \quad (20)$$

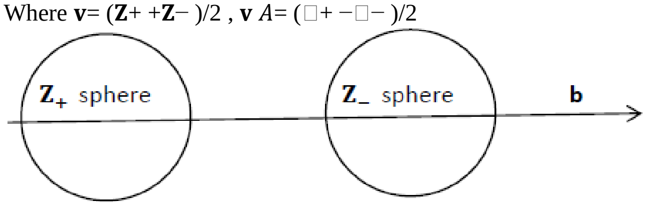
(3) The sum of all energy transfer rates along **v- b** are zero,i.e.,

$$S^{vb}(\alpha|\beta|\gamma) + S^{vb}(\alpha|\gamma|\beta) + S^{vb}(\beta|\alpha|\gamma) + S^{vb}(\beta|\gamma|\alpha) + S^{vb}(\gamma|\alpha|\beta) + S^{vb}(\gamma|\beta|\alpha) +$$
$$S^{bv}(\alpha|\beta|\gamma) + S^{bv}(\alpha|\gamma|\beta) + S^{bv}(\beta|\alpha|\gamma) + S^{bv}(\beta|\gamma|\alpha) + S^{bv}(\gamma|\alpha|\beta) + S^{bv}(\gamma|\beta|\alpha) = 0 \quad (21)$$

The Elsasser variable is an important parameter to describe MHD turbulence, which is defined as

$\mathbf{Z}\pm = \mathbf{v} \pm \Delta\mathbf{b}/\sqrt{4\pi\rho} = \mathbf{v} \pm \mathbf{v}\, \boldsymbol{A}$

Where $\mathbf{v} = (\mathbf{Z}+ + \mathbf{Z}-)/2$, $\mathbf{v}\, A = (□+ - □-)/2$



The Elsasser variables Z+ represent a wave travelling down the magnetic field and Z− represent a wave travelling up the magnetic field **b**, where V_A is the Alfven velocity.

The incompressible MHD equations in terms of Elsasser variables are

$$\frac{\partial \mathbf{Z}_\pm}{\partial t} \mp (\mathbf{v}_A.\nabla)\mathbf{Z}_\pm + (\mathbf{Z}_\mp.\nabla)\mathbf{Z}_\pm = -\nabla P + \nu_\pm \nabla^2 \mathbf{Z}_\pm + \nu_\mp \nabla^2 \mathbf{Z}_\mp \quad (22)$$

$\nabla.Z\pm = 0.$ (23)

Where $V_\pm = (V \pm \eta)/2$

Hence the energy transformed from Z± (β) to Z± (α) is given by

$$S^{++}(\alpha|\beta|\gamma) = -i\big(\alpha.\mathbf{Z}_-(\gamma)\big)\big(\mathbf{Z}_+(\alpha).\mathbf{Z}_+(\beta)\big) \quad (24)$$

$$S^{--}(\alpha|\beta|\gamma) = -i\big(\alpha.\mathbf{Z}_+(\gamma)\big)\big(\mathbf{Z}_-(\alpha).\mathbf{Z}_-(\beta)\big) \quad (25)$$

3. Shell to shell energy transfer in MHD turbulence

Since, There is no cross transfer of energy between Z_+ and Z_- modes so, $S^{+-}(\alpha|\beta|\gamma)$ and $S^{-+}(\alpha|\beta|\gamma)$ does not exist.

3. Shell to Shell energy transfer in MHD turbulence

Using the definition of mode to mode energy transfer, the energy transfer rate from the mth shell of the field Y to the □th shell of the field X is given by

$$T^{XY}_{lm} = \sum_{\alpha\epsilon l}\sum_{\beta\epsilon m} S^{XY}(\alpha|\beta|\gamma) \quad (26)$$

The α-sum is over the ιth shell and β-sum is over the mth shell. Where X and Y can be either **v** or **b** correspond to energy transfers between different shell variables. Hence the energy transfer rates in terms of velocity and magnetic field are

$$T^{vv}_{lm} = \sum_{\alpha\epsilon l}\sum_{\beta\epsilon m} S^{vv}(\alpha|\beta|\gamma) \quad (27)$$

$$T^{bb}_{lm} = \sum_{\alpha\epsilon l}\sum_{\beta\epsilon m} S^{bb}(\alpha|\beta|\gamma) \quad (28)$$

$$T^{vb}_{lm} = \sum_{\alpha\epsilon l}\sum_{\beta\epsilon m} S^{vb}(\alpha|\beta|\gamma) \quad (29)$$

$$T^{bv}_{lm} = \sum_{\alpha\epsilon l}\sum_{\beta\epsilon m} S^{bv}(\alpha|\beta|\gamma) \quad (30)$$

The Equations (26) to (30) satisfy the following property

$$T^{XY}_{lm} = -T^{YX}_{ml} \quad (31)$$

In terms of Elsasser variables, equation (26) can be written as

$$T^{++}_{lm} = \sum_{\alpha\epsilon l}\sum_{\beta\epsilon m} S^{++}(\alpha|\beta|\gamma) \quad (32)$$

$$T^{--}_{lm} = \sum_{\alpha\epsilon l}\sum_{\beta\epsilon m} S^{--}(\alpha|\beta|\gamma) \quad (33)$$

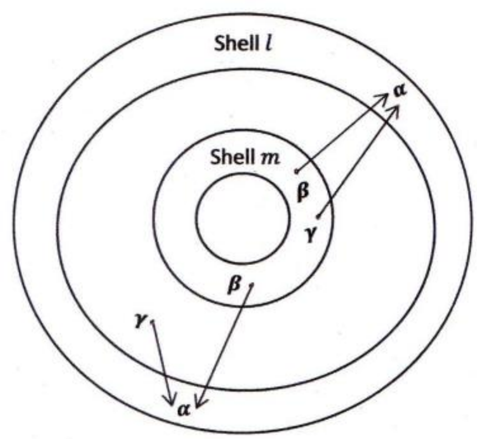


Fig.2: Shell to shell energy transfer from shell m to shell ι.

4. Energy Cascade rates in MHD turbulence

The energy transfer could take place from inside/outside of the v/b – sphere to the inside/outside of the v/b – sphere. In terms of $S^{YX}(\alpha|\beta|\gamma)$, the energy cascade rate from inside of the X- sphere of radius □0 to the inside of the Y- sphere of the same radius is

$$\prod\nolimits^{X<}_{Y<}(R_0) = \sum_{|\alpha|<R_0}\sum_{|\beta|<R_0} S^{YX}(\alpha|\beta|\gamma) \quad (34)$$

Here we denote the inside region of a sphere by < sign and outside region of a sphere by > sign. Where X and Y can be either **v** or **b**

There are various types of cascade rates in MHD turbulence. The energy flux from inside the **v**- sphere of radius R_0 to outside of the **v**- sphere is

$$\prod_{b>}^{v<}(R_0) = \sum_{|\alpha|>R_0}\sum_{|\beta|<R_0} S^{bv}(\alpha|\beta|\gamma) \quad (35)$$

The energy flux from inside the **b**- sphere of radius R_0 to outside of the **v**- sphere is

$$\prod_{v>}^{b<}(R_0) = \sum_{|\alpha|>R_0}\sum_{|\beta|<R_0} S^{vb}(\alpha|\beta|\gamma) \quad (36)$$

The energy flux from inside the **v**- sphere of radius R_0 to outside of the v- sphere is

$$\prod_{v>}^{v<}(R_0) = \sum_{|\alpha|>R_0}\sum_{|\beta|<R_0} S^{vv}(\alpha|\beta|\gamma) \quad (37)$$

The energy flux from inside the b- sphere of radius R_0 to outside of the b- sphere is

$$\prod_{b>}^{b<}(R_0) = \sum_{|\alpha|>R_0}\sum_{|\beta|<R_0} S^{bb}(\alpha|\beta|\gamma) \quad (38)$$

Using equations (35), (36), (37) and (38) the total energy flux is defined as

$$\prod_{tot}(R_0) = \prod_{b>}^{v<}(R_0) + \prod_{v>}^{b<}(R_0) + \prod_{v>}^{v<}(R_0) + \prod_{b>}^{b<}(R_0) \quad (39)$$

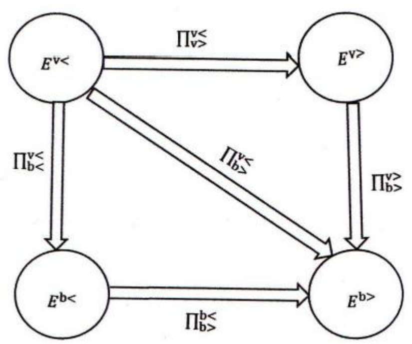


Fig.3: Various energy fluxes in Magneto hydrodynamic turbulence

In terms of Elsasser variables, equation (34) can be written as

$$\prod_{+>}^{+<}(R_0) = \sum_{|\alpha|>R_0}\sum_{|\beta|<R_0} S^{++}(\alpha|\beta|\gamma) \quad (40)$$

$$\prod_{->}^{-<}(R_0) = \sum_{|\alpha|>R_0}\sum_{|\beta|<R_0} S^{--}(\alpha|\beta|\gamma) \quad (41)$$

5. Conclusions

In the present paper, using the formalism of mode to mode energy transfer in magnetohydrodynamic (MHD) turbulence, two major conclusions are drawn . The first one is shell to shell energy transfer in MHD turbulence, i.e., T_{lm}^{XY}

Second is energy cascade rates (also called energy flux), i.e., $\prod_{Y>}^{X<}(R_0)$ in MHD turbulence.

With the help of Elsasser variables the shell to shell energy transfer is reduced in equations (32) and (33), and energy cascade rates are governed by the equations (40) and (41).

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